NOVIK TALK ON BIRATIONAL GEOMETRY NOTES

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Abstract. The classification of objects is a fundamental problem in any branch of mathematics. In birational geometry, we aim to study the classification of varieties under birational equivalence. Specifically, two varieties are birational if they are isomorphic everywhere except for a lower-dimensional locus. This property preserves many invariants, and it is well-known that in characteristic 0, every projective variety is birational to a smooth one. However, for a given variety, there can be several smooth varieties that are binational to the given one, leading to the natural question of whether we can select the simplest smooth variety in each birational equivalence class. This simplest variety is called a minimal model. This leads to the Minimal Model Program (MMP), which has been a central framework in birational geometry for many years.In this lecture, we will discuss the basic notions and frameworks of birational geometry and explore how the MMP works for surfaces and its generalization to higher-dimensional varieties. The termination of the MMP in higher dimensions remains an open question. This lecture aims to highlight the motivation and general spirit of birational geometry over fields of characteristic 0.

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1. Birational Geometry

Geometry over local fields is not very developed today. We will work with classical varieties, potentially projective, not schemes.

Definition 1.1. A **rational map** is a map $f : X \dashrightarrow Y$ consisting of pairs $\langle \phi, \phi_U \rangle$: $U \subseteq X$ an open set and $\phi_Y : U \to Y$ is an actual morphism such that $\langle U, \phi_U \rangle \sim \langle V, \phi_V \rangle : \phi_U|_{U\cap V} = \phi_V|_{U\cap V}$.

I.e. it is defined "almost" everywhere as open sets on varieties are typically large (dense).

Definition 1.2. A **birational map** is a rational map with a rational inverse (composition is true identity).

Say that two varities are birational if there is a birational map between them.

Birational geometry tries to classify varieties up to birationality. For example, we might want to distinguish smooth, non-singular, singular, etc. It is split up into 3 fields:

(1) Resolution of singularities: we want to find a map that preserves interesting properties to a non-singular variety
(2) Minimal model program (MMP): imagine we have a large birational variety class, take the smooth ones, and we want to pick the "best" one
(3) Rationality: a rational variety is a variety that is birational to projective space, which is nice because it allows us to locally introduce coordinate

Observation 1.3. Take two varieties, X, Y. The following are equivalent:
(1) X, Y birational
(2) $\forall U \subseteq X, V \subseteq Y$ s.t. $U \cong V$ with U, V open
(3) $K(X) \cong K(Y)$ as k-algebras

Observation 1.4. Take two projective, normal varieties, X, Y with a birational map $f : X \dashrightarrow Y$ between them. Then with $U \subseteq X$ the largest open set on which $X \setminus U$ is closed and $\operatorname{codim} X \setminus U \geq 2$.

This implies that two smooth birational projective curves are isomorphic because we can't find a subset of codimension larger than 2 in it.

Theorem 1.5 (Hironaka Resolution Theorem)**.** *For all projective varieties X over a characteristic 0 field, $\exists Y$ a smooth projective variety s.t. X, Y are birational.*

This is open for characteristic p with $\dim \geq 4$. The existence isn't as useful as actually having the construction. We have one!

1.1. **Blow up.** Imagine $\mathbb{A}^2$ with $(0,0)$ in it. Now to blow this up, we want to replace the point with the projective line and modify the plane such that it remains isomorphic to it while integrating with the line nicely i.e. a spiral. Formally, the plane is now $\{(x,y)[z : w] : xz = yz\} \subseteq \mathbb{A}^2 \subset \mathbb{P}$. Here we have the projection π that makes the rest of the plane isomorphic. Then $\pi^{-1}((0,0)) \cong \mathbb{P}$, i.e. an effective divisor.

In general, given a smooth projective surface S, (S' is the blowup) $\pi : S' \to S$ is a blow up of a point P $E = \pi^{-1}(P)$ is an exceptional divisor. Further, $E \cong \mathbb{P}^1$.

Exercise 1.1.1. $E^2 = E.E = -1$.

This negative self-intersection number implies that intuitively, we can't wiggle it at all (in the sense that blow ups introduce wiggle room to let us distinguish intersection points with multiplicity).

Now imagine $\mathbb{P}^2$ and a normal curve in it. Now we blow up this curve at its self-intersection.

Definition 1.6. Let C be an irreducible curve, S be a smooth projective surface. Then with, $\pi : S' \to S$ the blowup at a point P, call $\overline{\pi^{-1}(C \setminus P)}$ the **strict transformation**.

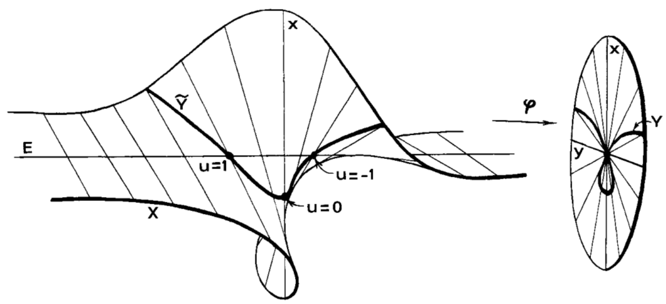
Theorem 1.7. *Given an irreducible curve $C \subseteq S$ a smooth projective surface, $\exists$ a sequence of blowups*

$$s_m \to \cdots \to S_0 = S$$

s.t. the strict transformation C_n of C is smooth at each level.

A nice property of this is that arithmetic genus drops with each step.

We can also classify the type of singularities: nodal ones have two tangent directions and cuspidals have one (maybe I misheard the number). For example, the



t≠0

Figure 3. Blowing up.

Figure 1. Nice picture of blowup from Hartshorne

blowup at the tip of a cone would become $\mathbb{P}^1$ because there are this many tangent vectors.

2. Surfaces

Imagine we start with a smooth surface S. The idea of Italian geometers of days past is that the definition of the nicest surface in a birational class can't be blown down. The idea of blowdown is to take a rational curve that with self-intersection equal to -1 and contract it to a point.

Proposition 2.1 (Castelnovo)**.** *Given a smooth projective surface S, there is a rational curve C with negative self intersection s.t. there is S_0 and $f : S \to S_0$ s.t. f is a blow up of a point with S_0 smooth.*

Definition 2.2 ("naive" definition of minimal surface)**.** A minimal surface is the one that doesn't have a -1 curve.

What about $\mathbb{P}^1 \times \mathbb{P}^1$? This is an exception, leading to more development of the theory. We really only want to consider varieties s.t. $\exists d$ with $\dim H^0(X, \omega_X^d) \neq 0$ $(K(X) \neq -\infty)$. In the case of surfaces, this exclusions ruled surfaces and $\mathbb{P}^2$.

Now what about higher dimensions? The above definition was good enough for surfaces, but not for curves.

Definition 2.3. We say that a smooth projective surface S is the minimal model if k_s, the canonical divisor is nef $(k_S.C \geq 0)$.

The canonical divisor is just a representation of the set of canonical bundles?

This is equivalent on surfaces to the above criterion. The proof is vaguely due to the adjunction formula: If k_s is nef $\implies k_S.C \geq 0 \forall$ irreducible C. If k_S is nef and there exists a -1 curve, then $k_S.E \geq 0$ for this -1 curve E as well. By realizing the canonical divisor on E as the canonical divisor on $\mathbb{P}^1$, $\deg k_E = (k_S + C).C$ Then $-2 = k_S.C + C^2 \geq -1$, a contradiction. Hence the minimal model has no -1 curves.

The other direction uses information theory?

For surfaces, MMP works well. Take a surface, contract -1 curves, to get S' and finish. This algorithm terminates because the Picard number drops. As the dimension increases, different kinds of singularities can arise.

Definition 2.4. Call X a projective variety a minimal model if k_X is nef and X has terminal singularities.

Take a smooth projective variety X has its canonical divisor nef, then we are done by above theorem. If it isn't, then there is C on X s.t. $k_X.C < 0$. By contracting it, we get a sequence

$$X \to X_1 \to \cdots .$$

But we aren't sure if this terminates nor do we know if contraction reduces the number of singularities or preserves the canonical divisor as a Cartier one. Instead, we replace $X_n \to X_{n+1}$ that is bad with "flip". The flip operation replaces the curve s.t. there are no more bad things.

We want to intuitively think about it as having a line in a plane and then rotating it. Now we have two questions:

(1) Why does flip exist
(2) Why does the flip terminate the sequence

The first question is yes, the second is open.