THE ECKMANN-HILTON ARGUMENT IN HIGHER DIMENSIONS

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1. Eckmann Hilton Argument on Sets

We want to make two monoidal structures compatible. This argument began with Ĉech, but other people built on his arguments to generalize abelian-ness.

Theorem 1.1 (EH Argument). *Given two monoid structures $\cdot, *$ on a set such that they have the same unit and they satisfy*

$$(a \cdot b) * (c \cdot d) = (a * c) \cdot (b * d),$$

then they are the same and commutative. I.e., if we have a monoid internal to the category of monoids, then they are the same. In fact, we don't even need associativity for monoid-ness.

As shorthand, write $a \cdot b = \frac{a}{b}$ and $a * b = a|b$. This is to suggest the idea of vertical and horizontal composition of cells. The distributivity is just $\frac{a}{b}|\frac{c}{d} = \frac{a|c}{b|d}$. This is more geometric looking.

Proof. We want to move $a|b$ to $b|a$. We do so via $a|b \to \frac{a}{1}|\frac{1}{b} \to \frac{a|1}{1|b} \to \frac{a}{b} \to \frac{1|a}{b|1} \to \frac{1}{b}|\frac{a}{1} \to b|a$. The other direction also works (move a down first).

$$a|b \to \frac{1}{a}|\frac{b}{1} \to \frac{1|b}{a|1} \to \frac{b}{a} \to \frac{b|1}{1|a} \to \frac{b}{1}|\frac{1}{a} \to b|a.$$

Notice that we did *vihhiv* where v is vertical unit, h is horizontal unit, and interchange is distributivity. □

2. Weak EH on Categories

Definition 2.1. Categorification from sets to categories:

set	$\to$	category
$=$	$\to$	$\cong$
monoid $\cdot, *$	$\to$	monoidal structure
interchange	$\to$	weak interchange
unit laws	$\to$	weak unit laws

This highlights the distinction between weak and strict equality: weak is up to the category, strict is true equality. For example, we have weak equalities $(A \times B) \times C \cong A \times (B \times C)$. In this way, we can weaken a monoidal structure to have $(a \otimes b) \otimes c \cong a \otimes (b \otimes c)$, $I \otimes a \cong a$, and $a \otimes I \cong a$ instead of equality.

The EH argument then becomes

Theorem 2.2. *Given two monoidal structures on the same category such that they are a weak monoid internal to* ***MonCat****.*

For a reference, Joyal and Street, BFSV, Aguiar Machigan, Guana are big names (I probably misspelled).

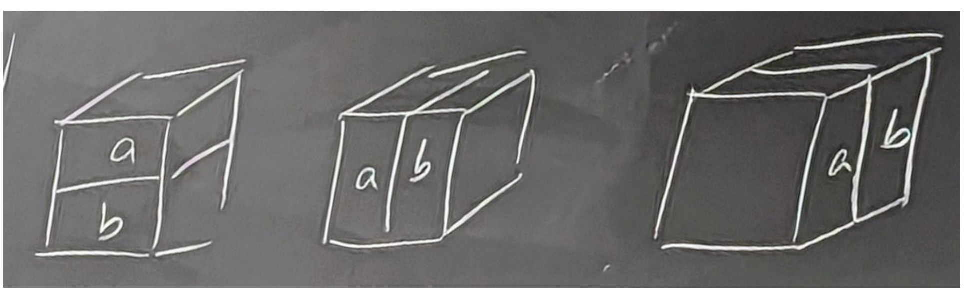
Corollary 2.3. *We instead have $a \otimes b \cong b \otimes a$. This is a* ***braiding*** *so that A becomes a braided monoidal category. This allows us to distinguish between the two ways of going $a|b \to b|a$: the braid crossings are different. The Young Baxter equation gives us a relation here (idk how to draw braids). If the crossings are the same, then this is called a symmetric monoidal category. This is equivalent to saying that doing the braiding twice gives us the identity.*

We originally believed interchanges needed to be strict. Eventually we Eugenia Chang et al weakened this to be weak. If we have strict associativity, then we can also do this.

3. Higher Order EH

Idea is that a third monoidal structure will force/enables the braiding to be a symmetry. We have interchange be strict here for simplicity.

Denote the three operations as follows:



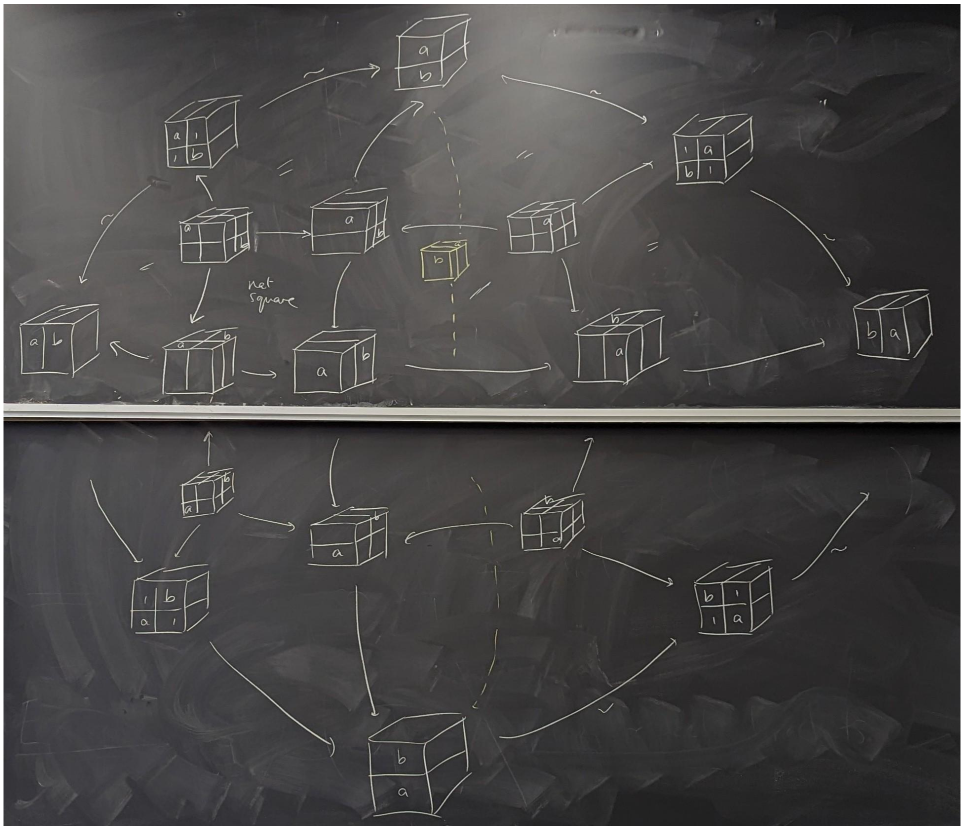
Theorem 3.1. *A weak monoid internal to duoidal categories is a symmetric monoidal category. We will take interchange to be strict to improve the diagrams.*

1x	monoid	monoidal category	monoidal 2-cat
2x	commutative monoid	braided monoidal category	braided monoidal 2-cat
3x	commutative monoid	symmetric monoidal category	sylleptic monoidal 2-cat
4x	commutative monoid	symmetric monoid category	

The table can be filled out more.

4. Relationship with Higher Dimensional Category Theory

This is related to HDCT due to the concept of degeneracy. We can reduce the number of dimensions by cutting off the top or the bottom. The starting idea for



this is the degenerate category has one object, i.e. a monoid as all compositions are composable.

A 1-object 2-category is a monoidal category is a monoidal category.

A 1-object 1 cell is a 2-category. Eckmann Hilton applies here to get that this is a commutative monoid.

The idea is that a k-degenerate $(n+k)$ category should be a k-monoidal n-category.