GEOMETRY

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Abstract.

Contents

1. 6/11

First consider $\mathbb{C}[t]$. What are the primes here? They are irreducible elements, which are degree 1 elements by the fundamental theorem of algebra.

Theorem 1.1 (FTA)**.** *Every polynomial $p(t)$ of degree > 0 in $\mathbb{C}[t]$ has a root.*

Corollary 1.2. *By Bezout's Lemma,*

$$p(t) = (t-\rho)q(t) + r(t) \qquad \deg r < \deg(t-\rho) = 1.$$

So r is a constant. Then letting $t = \rho$ gives us that $r = 0$. Repeating inductively gives us a unique prime factorization.

A classic proof is using complex analysis. Because $\frac{1}{|p(t)|} < C$, with Lioville's Theorem bounded continuous functions are constant, a contradiction. So it has a root.

Instead today we will use real analysis.

First though, we have another geometric proof. We want to show that if p is a polynomial of degree > 0, then $p : \mathbb{C} \to \mathbb{C}$ is onto.

Some call this the complex plane, others call it the complex line. Here it will be the complex plane.

First consider S^2, the boundary of the circle in $\mathbb{R}^3$. Put it on top of the plane in $\mathbb{R}^3$. Imagine we have evil Santa at the north pole. Evil Santa shoots points on the plane to points on S^2. We have good Santa at the origin and Santa at infinity. I.e. we wrap the plane around S^2.

So p gives a map from S^2 to S^2 by translating from $\mathbb{C}$ to S^2 with Santa. We can deal with infinity by adding a point on and considering $\frac{1}{p(\frac{1}{n})}$, which is smooth at 0 (i.e. changing coordinates). The induced map $\hat{p} : S^2 \to S^2$ is smooth.

Proposition 1.3. *Let f be the real part of $p(x+iy)$ and g the imaginary part of $p(x+iy)$. Then*

$$\det(\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = ||p'(z)||^2.$$

Proof. We can think of the Jacobian as a linear approximation to the action of $\mathbb{C}$ on $\mathbb{C}$ as a one dimensional $\mathbb{C}$ vector space, so it is multiplication. It is multiplication by $||p'(z)||^2$ because it stretches both dimensions over $\mathbb{R}$ by the same amount. $\square$

Theorem 1.4 (Inverse Function). *Given $f : U \to V$ for U, V open neighborhoods around 0 of $\mathbb{R}^2$. Then if Jac$(f) \neq 0$, there exists an inverse function g defined around 0 in the target.*

Definition 1.5. $v \in \mathbb{R}^2$ is a critical point if $\text{Jac}(f)(v) = 0$.

Because f is continuous, $(\text{Jac}(f))^{-1}(0)$ is closed, so the set of non-critical points is open. So $f(\Omega)$ is open whenever $\Omega^{open} \subset$ non critical point for reasons I didn't catch.

Proof of FTA. Consider $p : S^2 \to S^2$. The citical points of p is finite because p is apolynomial.

First we show that $p(S^2)$ is open. We do this by first seeing that $p(S^2 \setminus p^{-1}(p(criticalpoint)))$ is open via the above argument. There are at most $d(d-1)$ points that go to a critical value.

Further, the image is closed because the image of a compact set under a continuous image is compact and hence closed (under enough point set niceness). Hence $p(S^2 \setminus p^{-1}(p(criticalpoint)))$ is closed with some point set shenanigans.

We then have a contradiction because S^2 is connected, so p is surjective and we are done. $\square$

Now why doesn't this work for all smooth functions? For example, what about $(f(x,y), g(x,y)) : \mathbb{R}^2 \to \mathbb{R}^2$? The key fact is in the Jacobian calculation, allowing us to limit critical points.

We now use other related geometric ideas to show that an analogue of Fermat's Last Theorem fails.

Theorem 1.6 (FLT). *For $p > 2$, $x^p + y^p = z^p \implies xyz = 0$ for $x, y, z \in \mathbb{Q}$.*

Theorem 1.7 (FLT Analogue). *$x, y, z \in \mathbb{C}(t($, the rational polynomials in one variable $(p(t)/q(t))$. With $p = 2$, we have $(\frac{1-t^2}{1+t^2})^2 + (\frac{2t}{1+t^2})^2$ is a perfect square.*

Suppose there were $x(t), y(t)$ such that $x^p + y^p = 1$. This gives us a map $\mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by sending t to $(x(t), y(t))$. By making these into maps of spheres, we see that the image will be a multi-holed torus.

The map z^2 turns S^1 into a twisted doubled, S^1. If we attach a stick between two S^1, the preimage becomes a 2 holed torus. Increase z^p to have p-fold covers. But the preimages of points can't grow to infinity in number because of the numerator having finite degree, giving a fixed point (I think?) giving us a root.

To have more details, we have a bunch of p-fold covers giving us the diagram

$$\begin{tikzcd} & \hat{Y} \arrow[d] \\ Z \arrow[ur] \arrow[r] & Y \end{tikzcd}$$

Then for a point $z \in Z$, lift it up to $\hat{Y}$, and keep lifting.

If x is never 0, then each time we lift we get p points, unless we get ramification. I'm not sure what's going on past this point but we should end up getting that the image is a multiholed torus. Put the vertices of a triangulation on ramified points, then the number of edges and vertices multiply by p in the lift, but the number of points doesn't directly multiply by p. So the Euler characteristic goes really negative, giving it holes.