CREATION AND ANNIHILATION AS DIFFERENTIAL OPERATORS ON THE RIEMANN SPHERE

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Contents

1. Lecture 1

The title is literally false advertising. We will talk about raising and lowering operators, which are needed to understand the things in the title, but aren't exactly them.

1.1. Raising and Lowering Operators.

Fix a vector space over $\mathbb{C}$ and let h, e, f be linear operators on V. They interact as follows:

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h$$

where $[a, b] = ab - ba$, the commutator. Commutators measure how much two elements fail to commute, in the sense that if they commute, the commutator is 0.

Example 1.1. Let $V = \mathbb{C}^2$. Then with

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{1.2}$$

they span

$$\mathfrak{sl}_2(\mathbb{C}) = \{A \in \mathrm{Mat}_2(\mathbb{C}) | \operatorname{tr} A = 0\}.$$

We write $\mathfrak{sl}_2$ as shorthand.

We have the following isomorphism:

$\{$operators $\overline{h}, \overline{e}, \overline{f}$ on V satisfying Equation (1.2)$\}$ $\cong$ $\{$linear maps $\phi : \mathfrak{sl}_2 \to \mathrm{End}_{\mathbb{C}}(V)$ satisfying $\phi([A,B]) = [\phi(A), \phi(B)]\}$.

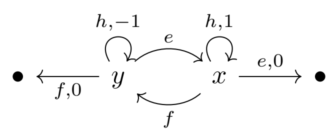
The backwards direction is that $\phi \mapsto \overline{h} = \phi(h), \overline{e} = \phi(e), \overline{f} = \phi(f)$. The bottom is are the $\mathfrak{sl}_2$-representations on V.

Note that the commutator relation is not associative. Instead it satisfies the Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

1.2. **Finite Dimensional Representations.** We're going to classify all finite dimensional representations. The infinite dimensional ones are subtle.

Definition 1.3. Let $V^1 := \mathbb{C}^2 = \mathbb{C}x \oplus \mathbb{C}y$ be the **standard representation**. The group action can be as follows: $hx = x, hy = -y, ex = 0, ey = x, fx = y, fy = 0$. We have the following diagram:



Example 1.4. Let $\lambda \in \mathbb{Z}, \lambda \geq 0$ and

$$V^\lambda := \mathrm{Sym}^\lambda(V^1) = \bigoplus_{k=0}^{\lambda} \mathbb{C}x^{\lambda-k}y^k.$$

Then $\dim V^\lambda = \lambda + 1$. We want $\mathfrak{sl}_2 \circlearrowright V^\lambda$ by extending $\mathfrak{sl}_2 \circlearrowright V^1$ using the Leibniz rule. In abstract algebra, we typically extend linearly by acting on x, y individually. But we want the action to act like differentiation because $\mathfrak{sl}_2$ is sort of the "derivative" group of $\mathbf{SL}_2$. So

$$(A \in \mathfrak{sl}_2, f, g \in \mathrm{Sym}(V^1)) \qquad A(fg) = (Af)g + f(Ag)$$

i.e. the Leibniz rule.

We have a trivial representation $V^0 = \mathbb{C}$. Here, $A1 = A(1\cdot 1) = (A1)\cdot 1 + 1\cdot (A1) = A1 + A1 \implies A1 = 0$. This is the trivial action.

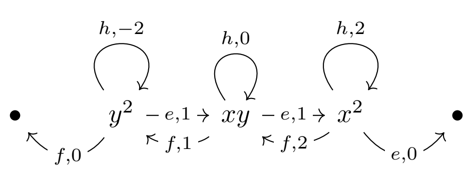
Example 1.5. We have

$$B^2 = \mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2 \cong \mathfrak{sl}_2.$$

This is because x^2, xy, y^2 are like e, f, h. The action is

$$\begin{aligned} hx^2 &= (hx)x + x(hx) = x^2 + x^2 = 2x^2 \\ hxy &= (hx)y + x(hy) = xy - xy = 0 \\ hy^2 &= (hy)y + y(hy) = -2y^2 \end{aligned}$$

This is called the adjoint representation. Further, $ex^2 = 0, exy = x^2, ey^2 = 2xy$. The operator f is the lowering operator, so $fx^2 = xy, fxy = y^2, fy^2 = 0$. The diagram is



In the general case, $\lambda \in \mathbb{Z}$, $V^\lambda = \bigoplus_{k=0}^{\lambda} \mathbb{C}x^{\lambda-k}y^k$. The action is $hx^{\lambda-k}y^h = (\lambda-2k)x^{\lambda-k}y^k$. We expect e to raise the power of x, and $ex^{\lambda-k}y^k = kx^{\lambda-k+1}y^{k-1}$ and f to lower, so $fx^{\lambda-k}y^k = (\lambda-k)x^{\lambda-k+1}y^{k+1}$.

Notice that h acts diagonalizably on V^λ with eigenvalues (all multiplicity one) $\lambda, \lambda-2, \lambda-4, \cdots, -\lambda+2, -\lambda$. For all h eigenvector with eigenvalue μ, we have the following: For $v \in V^\lambda$, $ev = 0 \iff \lambda = \mu$; otherwise ev is an h-eigenvector with eigenvalue $\mu + 2$. Likewise, $fv = 0 \iff \mu = -\lambda$; otherwise fv is an h-eigenvector with eigenvalue $\mu - 2$.

Exercise 1.2.1. Prove that V^λ is irreducible as an $\mathfrak{sl}_2$ representation $\forall \lambda \geq 0$. (Hint: any non-zero subrepresentation is in particular stable under the action of h, hence has an h-eigenvector).

Theorem 1.6. *Finite dimensional $\mathfrak{sl}_2$ representations can be classified as follows:*

(1) any finite dimensional irreducible $\mathfrak{sl}_2$ representation is isomorphic to V^λ
(2) any finite dimensional $\mathfrak{sl}_2$ representation is isomorphic to a direct sum of irreducibles

These representations come up in the theory of spin. The spin is half of the eigenvalue of h.

Proof sketch of 1: The weights on V, a $\mathfrak{sl}_2$ representation are the h-eigenvalues. A highest weight vector is an h-eigenvector v s.t. $ev = 0$. For example, V^λ has the highest weight vector x^λ, but one could take a direct sum of two different V^λ, V^μ and have highest weight vectors that aren't the largest h-eigenvalue.

Step 1: $\dim V < \infty \implies V$ contains a highest weight vector.

Step 2: $\dim V < \infty, V$ irreducible $\implies V$ has a unique highest weight vector up to scaling.

Step 3: $\dim V < \infty, V$ irreducible, V has unique highest weight $\lambda \implies V \cong V^\lambda$.

For ii: one can get an action of $\boldsymbol{SU}_2$ on the group, which is compact, allowing us to do the averaging trick. This is sort of analytic.

Next time we will talk about highest weight representations.

Definition 1.7. V is a highest weight representation of $\mathfrak{sl}_2$ if

(1) V is finitely generated as $\mathfrak{sl}_2$ representations, i.e. there is a finite set of vectors such that all representations containing it are V.
(2) $h \circlearrowleft V$ diagonalizably;
(3) $e \circlearrowleft V$ locally nilpotently, i.e. $\forall v \in V, e^n v = 0 \forall n >> 0$ (n can depend on v).

A typicaly highest weight infinite dimensional representation has the operation diagram of V^1, but infinitely to the left.

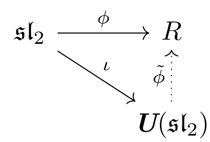
2. Lecture 2

Addendum: the sketch of classifying finite dimensional $\mathfrak{sl}_2$ representations, we have to check that, for finite dimensional representations, the weights are integral. This is something unique to the finite dimensional representations, as the eigenvalues of h are integral.

2.1. Universal Enveloping Algebra.

Claim 2.1. $\exists$ *associative $\mathbb{C}$-algebra $\boldsymbol{U}(\mathfrak{sl}_2)$ (i.e. a ring with $\mathbb{C}$ in its center) satisfying the following properties:*

(1) $\exists$ *injective* $\iota : \mathfrak{sl}_2 \hookrightarrow \boldsymbol{U}(\mathfrak{sl}_2)$
(2) $\iota([A,B]) = [\iota(A), \iota(B)] \forall A, B \in \mathfrak{sl}_2$
(3) *a universal property: for any associative* $\mathbb{C}$*-algebra with* $\phi : \mathfrak{sl}_2 \to \mathbb{R}$ *that is* $\mathbb{C}$*-linear s.t.* $\phi([A,B]) = [\phi(A), \phi(B)]$ *for all* $A, B \in \mathfrak{sl}_2$*, then there is a unique* $\mathbb{C}$*-algebra homomorphism* $\tilde{\phi} : \boldsymbol{U}(\mathfrak{sl}_2) \to R$ *s.t.* $\tilde{\phi} \circ \iota = \phi$*, i.e.*



Exercise 2.1.1. Formulate and prove uniqueness of $\mathbf{U}(\mathfrak{sl}_2)$.

Remark 2.1. Note that $\mathrm{Mat}_2(\mathbb{C})$ satisfies (i-ii) but not (iii). In a sense, this ring is larger than the universal enveloping algebra.

Proof of Existence. We start with the tensor algebra:

$$T(\mathfrak{sl}_2) := \bigoplus_{d \geq 0} \mathfrak{sl}_2^{\otimes d}.$$

This is a free associative algebra on $\mathfrak{sl}_2$ as a vector space, but might not follow property (ii). This, like a polynomial ring has a basis consisting of monomials, e.g. $e^2 \otimes h \otimes f^3 \otimes h^4 \otimes f$ (note that terms don't commute). Now let $I \subseteq T(\mathfrak{sl}_2)$ be the two-sided ideal generated by $A \otimes B - B \otimes A - [A,B]$. Now let $\mathbf{U}(\mathfrak{sl}_2) := T(\mathfrak{sl}_2)/I$. This forces the relation $A \otimes B - B \otimes A = [A,B]$, i.e. $[\iota(A), \iota(B)] = \iota(A,B)$ with $\iota : \mathfrak{sl}_2 \hookrightarrow T(\mathfrak{sl}_2) \twoheadrightarrow \mathbf{U}(\mathfrak{sl}_2)$. $\square$

Theorem 2.2 (Poincaré-Birkhoff-Witt)**.** *The monomials* $h^i e^j f^k (i,j,k \geq 0)$ *form a basis for* $U(\mathfrak{sl}_2)$*. Same for any other ordering. I.e. we have a* $\mathbb{C}$*-linear isomorphism (but not an associative algebra isomorphism)* $\boldsymbol{U}(\mathfrak{sl}_2) \cong \mathrm{Sym}(\mathfrak{sl}_2)$*.*

This is true for any lie algebra as well.

Finally, the benefit of this is that V a $\mathfrak{sl}_2$ representation $\iff$ $\mathfrak{sl}_2 \to \mathrm{End}_{\mathbb{C}}(V)$ $\iff$ there is a unique associative algebra homomorphism $U(\mathfrak{sl}_2) \to \mathrm{End}_{\mathbb{C}}(V)$, giving V a $U(\mathfrak{sl}_2)$ module structure.

So $\{\mathfrak{sl}_2 -$ representation structures on $V\} \cong \{U(\mathfrak{sl}_2 -$ module structures on $V\}$.

Corollary 2.3 (Casimir Operator)**.** *The product of things in the Lie algebra isn't closed. But the RHS is. The Casimir Operator helps us combine these ideas.*

$$\Omega := \frac{1}{2}h^2 + ef + fe \in U(\mathfrak{sl}_2).$$

We have that $Z(\mathfrak{sl}_2) := Z(U(\mathfrak{sl}_2)) = \{x \in U(\mathfrak{sl}_2) | xy = yx \forall x \in U(\mathfrak{sl}_2)\}$*, i.e. the center.*

Exercise 2.1.2. $\Omega \in Z(\mathfrak{sl}_2)$.

Remark 2.4. $Z(\mathfrak{sl}_2) \cap \mathfrak{sl}_2 = \{0\}$, giving the universal enveloping algebra more nice properties (namely having a non-zero center).

A common technique when studying an algebra is to compute the center, which is hopefully large, and study modules over the center.

Theorem 2.5 (Harish-Chandra)**.** $Z(\mathfrak{sl}_2) = \mathbb{C}[\Omega]$.

2.2. **Highest Weight Representations.** Let V be a $\mathfrak{sl}_2$ representation.

Definition 2.6. Call V a **highest weight representation** if

(1) V is finitely generated as a $\mathfrak{sl}_2$-representation, which with our new tool of universal enveloping algebra, it is just that V is f.g. as a $U(\mathfrak{sl}_2)$-module.
(2) h acts diagonalizably on V, which is motivated by this being a property of finite dimensional ones
(3) e acts locally nilpotently on V (this allows us to manufacture highest weight vectors)

e.g. any finite-dimensional representation is highest weight. The condition i) is met easily, and ii-iii) follow from the classification of finite dimensional representations as a direct sum of V^λ.

Example 2.7 (Verma)**.** Non-finite dimensional representation:

$$\mathfrak{G} := \mathbb{C}h + \mathbb{C}e = \left\{ \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix} \mid a, b \in \mathbb{C} \right\} \subseteq \mathfrak{sl}_2$$

which is stable under $[-,-]$.

Now we are going to use the same technique as for $\mathbf{SL}_2(\mathbb{F}_q)$: the principal series.

Let $C^\lambda := \mathbb{C}$ with $\mathfrak{G}$-action given by $h1 = \lambda, e1 = 0$. This then makes $\mathbb{C}^\lambda$ a left module over $U(\mathfrak{G})$.

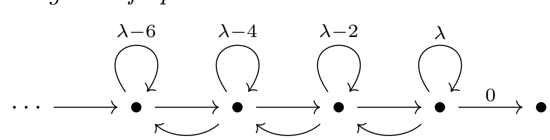
Definition 2.8 (Verma Module)**.** $M^\lambda := U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{G})} \mathbb{C}^\lambda$. A priori, this is just a vector space, but the action of $U(\mathfrak{sl}_2)$ via multiplying on the left makes it a $U(\mathfrak{sl}_2)$ module, making it a $\mathfrak{sl}_2$ representation.

By Theorem 2.2, $M^\lambda \cong \mathbb{C}[f]$ as a vector space via $f^n \otimes 1 \mapsto f^n$. Hence M^λ is infinite dimensional. Now $\forall k \in \mathbb{Z}^{\geq 0}$, let $v_{\lambda-2k} := f^k \otimes 1 \in M^\lambda$. Hence $M^\lambda = \bigoplus_{k \geq 0} \mathbb{C}v_{\lambda-2k}$.

Proposition 2.9.

(1) $fv_{\lambda-2k} = v_{\lambda-2k}$
(2) $hv_{\lambda-2k} = (\lambda - 2k)v_{\lambda-2k}$
(3) $ev_{\lambda-2k} = k(\lambda - k + 1)v_{\lambda-2k+2}$.

Now we have this diagram of operations



Corollary 2.10. *M^λ is irreducible $\iff \lambda \notin \mathbb{Z}^{\geq 0}$.*

Proof. If $\lambda \in \mathbb{Z}^{\geq 0}$, then M^λ fits into a SES

$$0 \to M^{-\lambda-2} \to M \to V^\lambda \to 0$$

i.e. the image of a map equals the kernel of the map after it.

This doesn't split, i.e. $M^\lambda \ncong V^\lambda \oplus M^{-\lambda-2}$. This is our first example of a non-semisimple module. □

Finally, we classify the decompositions:

$$L^\lambda := \begin{cases} V^\lambda & \text{if } \lambda \in \mathbb{Z}^{\geq 0} \\ M`\lambda & \text{if } \lambda \notin \mathbb{Z}^{\geq 0} \end{cases}.$$

So $\lambda \neq \mu \implies L^\lambda \neq L^\mu$.

Corollary 2.11. *Any irreducible highest weight $\mathfrak{sl}_2$ representation is isomorphic to L^λ for a unique $\lambda \in \mathbb{C}$.*

But non-semisimple highest weight representations exist because representations might not decompose into a direct sum of irreducibles. Now we have to classify indecomposable representations. The Vermas modules are a good example as they have a non-trivial subrepresentation, but doesn't split as a direct sum.

Now let $\mathscr{O} = \{\text{highest-weight } \mathfrak{sl}_2 \text{ representations}\}$, the "BGG category $\mathscr{O}$". Let $\mathscr{O}_0 = \{V \in \mathscr{O} | \Omega \text{ acts locally nilpotently on } V\}$. Because Ω acts locally finitely on V, Ω can decompose things nicely. Here Ω has only eigenvalues 0.

Theorem 2.12. *$\mathscr{O}_0$ contains exactly 5 indecomposables:*

$$L^0 = \mathbb{C}^0, L^{-2}, M^0, (M^0)^V, P^{-2}.$$

The first two are irreducible. The fourth is the dual Verma, and duality preserves irreducibility. The P^{-2} is the "big projective". It has a composition series of length 3:

$$0 \to M^0 \to P^{-2} \to L^{-2} \to 0.$$

Recall that M^0 has L^0, L^{-2} in it, so P^{-2} has a decomposition series with L^0, L^{-2}, L^{-2}.

3. Lecture 3

3.1. **Block Decomposition.** $Z(U(\mathfrak{sl}_2)) \cong \mathbb{C}[\Omega]$ with Ω the Casimir operator $\frac{1}{2}h^2 + ef + fe$. Denote $\mathscr{O}$ for the set of highest weight $\mathfrak{sl}_2$-representations. Because $U(\mathfrak{sl}_2)$ is not semisimple, we don't have decompositions into irreducibles. But we have decompositions into indecomposables.

Before, we restricted to locally nilpotent actions by the Casimir operator. This is fine though because $V \in \mathscr{O} \implies \Omega$ acts locally finitely on V. I.e., $\forall v \in V, \operatorname{span}(v, \Omega v, \Omega^2 v, \Omega^3 v, \cdots)$ is finite dimensional.

Proof. This span is $\mathbb{C}[\Omega]v$. This means that $V \cong \oplus V_{[\lambda]}$ of generalized eigenspaces of Ω.

Let $s \cdot \lambda = -\lambda - 2$, i.e. s is like reflection over -1. So $\mathbb{C}/\langle s \rangle \cong \mathbb{C}$ via $\lambda \mapsto \frac{1}{2}\lambda(\lambda+2)$ except at -1. This is because $\Omega \circlearrowright L^\lambda$ by scalar $\frac{1}{2}\lambda(\lambda+2)$. Similarly, $\Omega \circlearrowright L^{-\lambda-2}$ by scalar $\frac{1}{2}\lambda(\lambda+2)$. The index of the decomposition is over $[\lambda] \in \mathbb{C}/\langle s \rangle$ and $V_{[\lambda]}$ the generalized eigenspace of eignevalue $\frac{1}{2}\lambda(\lambda+2)$.

Finally, define $\mathscr{O}_{[\lambda]} = \{V \in \mathscr{O} | V = V_{[\lambda]}\}$ which has irreducibles $L^\lambda, L^{-\lambda-2}$ except at $\lambda = -1$ which has multiplicity 2. □

Now last time, we said that $\mathscr{O}_{[0]}$ contains 5 indecomposables. We have L^0, L^{-2}, $M^0, (M^0)^*, P^{-2}$ with the first two being irreducible. If $\lambda \in \mathbb{Z}^{\geq 0}$, then $\mathscr{O}_{[\lambda]} \cong \mathscr{O}_{[0]}$. Thus $\lambda \in \mathbb{C} \setminus \mathbb{Z}^{\geq 0} \implies \mathscr{O}_{[\lambda]}$ is semisimple.

3.2. **The Projective Line.** We are always over $\mathbb{C}$.

Definition 3.1. $\mathbb{P}^1 = \{\text{one dimensional subspaces of } \mathbb{C}^2\}$. We view $\mathbb{P}^1$ as an algebraic variety over $\mathbb{C}$. I.e., we only consider polynomial functions instead of homomorphic.

Let $\mathbb{A}^n = \mathbb{C}^n$ as a variety. Then $\mathbf{Fun}(\mathbb{A}^n) = \mathbb{C}[x_1, \dots, x_n]$. We can map $\mathbb{A}^2 \setminus \{0\}$ to $\mathbb{P}^1$ via $(x, y) \mapsto [x : y] := \mathbb{C} \cdot (x, y)$ (as a line). The square brackets are called homogenous coordinates.

We also have inhomogenous coordinates via $\mathbb{A}^1_0 := \{[x : y] | x \neq 0\} \subsetneq \mathbb{P}^1$, i.e. $\mathbb{A}^1_0 = \mathbb{P}^1 \setminus \{\infty\}$ with $\infty = [0 : 1]$. The local coordinates are

$$\mathbb{A}^1 \cong \mathbb{A}^1_0 : t \mapsto [1 : t].$$

So $\mathbb{P}^1 = \mathbb{A}^1_0 \cup \{\infty\}$. But we also have a different local coordinate system:

$$\mathbb{A}^1_\infty = \{[x : y] | y \neq 0\} = \mathbb{P}^1 \setminus \{0\}$$

with $0 = [1 : 0]$. Then $\mathbb{P}^1 = \mathbb{A}^1_0 \cup \mathbb{A}^1_\infty$ each with different coordinates, and on their intersection, their coordinates are reciprocal. Let t be the coordinates on $\mathbb{A}^1_0$ and s be the coordinates on $\mathbb{A}^1_\infty$.

Claim 3.1. *$\textbf{\textit{Fun}}(\mathbb{P}^1)$ only has constant global functions. Algebraically, this is saying that $f \in \textbf{\textit{Fun}}(\mathbb{P}^1)$, then restricted to the two affine parts above, we have $f|_{\mathbb{A}^1_0} = \sum_{n \geq 0} a_n t^n$ and $f|_{\mathbb{A}^1_0 \cap A^1_\infty} = \sum_{n \geq 0} a_n s^{-n}$. But f has no poles, so $a_n = 0, n > 0$.*

3.3. **Vector Fields.** We have that $\mathbf{Vect}(\mathbb{A}^1) = \mathbb{C}[t] \frac{d}{dt}$.

Claim 3.2. *We have $\frac{d}{dt}, t\frac{d}{dt}, t^2\frac{d}{dt} \in \textbf{\textit{Vect}}(\mathbb{A}^1_0)$ extend uniquely to vector fields to $\mathbb{P}^1$. In fact, they form a basis for $\textbf{\textit{Vect}}(\mathbb{P}^1)$.*

Proof. By the chain rule,

$$\frac{d}{dt} = \frac{ds}{dt} \cdot \frac{d}{ds} = -\frac{1}{t^2}\frac{d}{ds} = -s^2\frac{d}{ds}.$$

Hence $\xi \in \mathbf{Vect}(\mathbb{P}^1) \implies \xi \mid_{\mathbb{A}^1_0} = \sum_{n \geq 0} a_n t^n \frac{d}{dt}$ and $\xi \mid_{\mathbb{A}^1_0 \cap \mathbb{A}^1_\infty} = \sum_{n \geq 0} a_n s^{-n}(-s^2)\frac{d}{ds} = -\sum_{n \geq 0} a_n s^{2-n} \frac{d}{ds}$. Thus $a_n = 0$ for $n > 2$. This forms a basis of $\mathbf{Vect}(\mathbb{A}^1_\infty)$. □

We want to develop a Lie bracket to capture this differential.

3.4. **Differential Operators.** $\mathbf{Diff}(\mathbb{A}^1) = \oplus_{n \geq 0} \mathbb{C}[t] \frac{d^n}{dt^n}$. We can see that this is an associative $\mathbb{C}$-algebra via the key relation via composition of operators. I.e., we have

$$\left[\frac{d}{dt}, f(t)\right] = f'(t)$$

where this is the commutator bracket. A point of potential confusion is that $f(t)$ is the operator of multiplying by $f(t)$, not applying the function $f(t)$. I.e., $\frac{d}{dt}(fg) - f\frac{dg}{dt} = \frac{df}{dt}g$. This algebra is non-commutative.

Question 3.1. What is $\mathbf{Diff}(\mathbb{P}^1)$?

We can restrict via an injection (injection due to density of affine space) $\mathbf{Diff}(\mathbb{P}^1) \hookrightarrow \mathbf{Diff}(\mathbb{A}^1_0)$ via $\xi \mapsto \xi|_{\mathbb{A}^1_0}$. The image of this isn't very easy to described explicitly. For example, consider $t^4\frac{d^2}{dt^2} = \frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds}$, which doesn't extend to $\mathbb{P}^1$ as $\frac{1}{s}$ has a pole. Another weird example is $t^4\frac{d^2}{dt^2} + 2t^3\frac{d}{dt} = \frac{d^2}{ds^2}$ does extend.

Remark 3.2. If we have $\xi, \eta \in \mathbf{Vect}(\mathbb{P}^1) \subseteq \mathbf{Diff}(\mathbb{P}^1)$, then $[\xi, \eta] \in \mathbf{Vect}(\mathbb{P}^1)$.

Now identify $\mathfrak{sl}_2 \cong \mathbf{Vect}(\mathbb{P}^1)$ via $h \mapsto -2t\frac{d}{dt}$, $e \mapsto \frac{d}{dt}$, and $f \mapsto -t^2\frac{d}{dt}$.

Exercise 3.4.1. Show that this isomorphism respects the respective brackets.

The reason that $\mathfrak{sl}_2$ appears here is that $\mathbf{SL}_2$ acts on $\mathbb{P}^1$. The above isomorphism of Lie algebras makes $\mathfrak{sl}_2$ a tangent vector like thing on $\mathbb{P}^1$.

We have

$$\begin{array}{ccc} \mathfrak{sl}_2 & \xrightarrow{\sim} & \mathbf{Vect}(\mathbb{P}^1) \\ \downarrow & & \downarrow \\ U(\mathfrak{sl}_2) & \overset{\phi}{\dashrightarrow} & \mathbf{Diff}(\mathbb{P}^1) \end{array}$$

Exercise 3.4.2.

Claim 3.3. $\phi(\Omega) = 0$

Proof. Exercise! □

Define $U(\mathfrak{sl}_2)_0 := U(\mathfrak{sl}_2)/\Omega U(\mathfrak{sl}_2)$. Thus we have a homomorphism $\overline{\phi} : U(\mathfrak{sl}_2)_0 \to \mathbf{Diff}(\mathbb{P}^1)$.

Theorem 3.3 (Beilinson-Bernstein)**.** *$\overline{\phi}$ is an isomorphism.*

Proof Sketch. Both sides of this are non-negatively filtered as rings: degree on the left, n-th derivative on the right. Due to this, it suffices to show that there is an isomorphism on the graded level (note that $\overline{\phi}$ respects the grading), i.e.

$$\operatorname{gr}\overline{\phi} : \operatorname{gr} U(sl_2)_0 \cong \operatorname{gr}\mathbf{Diff}(\mathbb{P}^1).$$

Because they commute up to lower order terms, focusing on one grading makes them commutative. We have $\operatorname{gr}\mathbf{Diff}(\mathbb{P}^1) \hookrightarrow \mathbf{Fun}(T^*\mathbb{P}^1)$. Further, we have $\operatorname{gr} U(\mathfrak{sl}_2)_0 \cong \mathbf{Fun}(\mathcal{N})$ for a variety $\mathcal{N}$. If we have an isomorphism $\mathbf{Fun}(\mathcal{N}) \to \mathbf{Fun}(T^*\mathbb{P}^1)$, then we are done. I.e.,

$$\begin{array}{ccc} \operatorname{gr} U(\mathfrak{sl}_2)_0 & \xrightarrow{\sim} & \operatorname{gr}\mathbf{Diff}(\mathbb{P}^1) \\ \downarrow\wr & & \downarrow \\ \mathbf{Fun}(\mathcal{N}) & \xrightarrow{\sim} & \mathbf{Fun}(T^*\mathbb{P}^1) \end{array}$$

Call this bottom map $*$.

Let $\mathcal{N} = \{A \in \mathfrak{sl}_2 | A^2 = 0\}$. I.e., $\left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \middle| a^2 + bc = 0 \right\}$. This is the nilpotent cone (stable under scaling). We have that $T^*\mathbb{P}^1 \cong \{(L, A) | L \in \mathbb{P}^1, A \in \mathcal{N}, A(\mathbb{C}^2) \subseteq L\}$ (cotangent bundle).

We have that $*$ is pullback along $\mu : T^*\mathbb{P}^1 \to \mathcal{N}$ that sends $(L, A) \mapsto A$, i.e. the moment map.

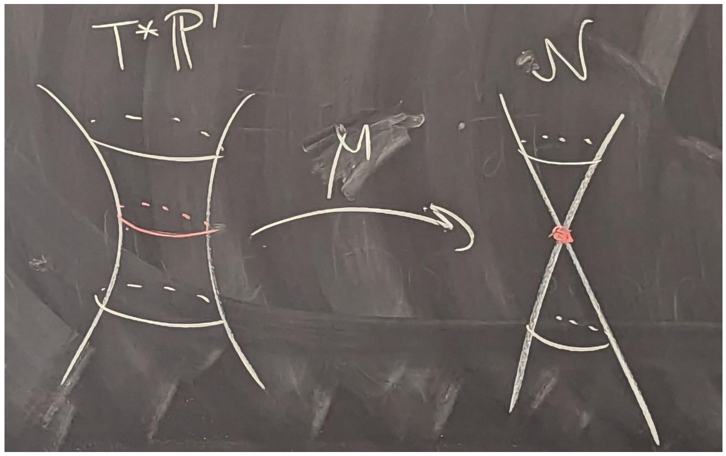
Observation 3.4. (1) $\mu^{-1}(0) \cong \mathbb{P}^1$
(2) $\mu : T^*\mathbb{P}^1 \setminus \mu^{-1}(0) \cong \mathcal{N} \setminus \{0\}$
(3) μ is a proper map (i.e. preimage of compact sets are compact).

These observations imply that this is an isomorphism.
This map looks like blowup. □

4. Lecture 4

We will be thinking of representations as sheaves.

4.1. **What is a sheaf?**



4.1.1. *Categorification.* This is a non-formal concept.

At the lowest abstraction is a number. Then vector space. Then category, and possible 2-category, and etc.

We can sort of go backwards: vector spaces have dimension, a category has a Grothendieck space, K_0.

Another non-abstract object is a function, which is a thing assigning numbers to numbers. A level up from this is a sheaf, which assigns vector spaces to each point.

Definition 4.1. Let X be a space (e.g. varieties, smooth manifolds, etc.). A **sheaf** $\mathscr{F}$ on X is something that takes a point $x \in X$ and maps it to $\mathscr{F}_x \in \mathbf{Vect}$. This is called the **fiber**.

The slogan is that sheaves are the categorification of functions. We can assign to X $\mathbf{Shv}(X) = \{\text{a flavor of sheaves on } X\}$.

Now take $f : X \to Y$ a map.

Definition 4.2. We have the **inverse sheaf** $f^* : \mathbf{Shv}(Y) \to \mathbf{Shv}(X)$ the inverse image defined by

$$(f^*\mathscr{F})_x = \mathscr{F}_{f(x)}.$$

Example 4.3. Subsheafs are the inverse sheaf of inclusion.

Definition 4.4. Let $f_* : \mathbf{Shv}(X) \to \mathbf{Shv}(Y)$ be the **direct image** sheaf, and think of it like integration along the fibers of f, which isn't perfectly defined all the time as with the inverse image. Sheaves are analogous to distributions (which are always integrable along fibers).

Example 4.5. Take $\rho : X \to pt$, $\mathscr{F} \in \mathbf{Shv}(X)$ produces $\rho_*\mathscr{F} \in \mathbf{Vect}$ which is like $\int_X \mathscr{F}$.

The key property that we want with these is that

Property 4.6.

$$\mathrm{Hom}_{\mathbf{Shv}}(f^*\mathscr{F}, \mathscr{G}) = \mathrm{Hom}_{\mathbf{Shv}}(\mathscr{F}, f_*\mathscr{G}).$$

We have some exceptional functors:

$$f_! : \mathbf{Shv}(X) \rightleftarrows \mathbf{Shv}(Y) : f^!.$$

This satisfies

$$\mathrm{Hom}_{\mathbf{Shv}(Y)}(f_!\mathscr{G}, \mathscr{F}) = \mathrm{Hom}_{\mathbf{Shv}(X)}(\mathscr{G}, f^!\mathscr{F}).$$

This is special because we have a reverse direction nice map.

We have notions of proper (preimage of compact is compact, e.g. closed embedding) where $f_! = f_*$ and open embedding where $f^! = f^*$.

Another key notion is base change:

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

The key property here is that

$$g^! f_* \cong f'_*(g')^! \qquad g^* f_! \cong f'_!(g')^*.$$

Example 4.7. Say we have an open embedding, $j : U \to X$. Then $j_!\mathscr{F}$ is the "extension by 0". We have

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle i} \\ U & \xrightarrow{j} & X \end{array}$$

Can also see an exercise in Hartshorne Chapter 2 section 1. We have the relation $i^* j_! \mathscr{F} = 0$.

Technically, everything here needs to be derived. This adjective means that instead of vector spaces, we instead work with complexes.

4.2. **Quasicoherent Sheaves.** Define

$$\mathbf{QCoh}(\mathbb{A}^1) \coloneqq \mathbb{C}[t] - \text{modules} = \mathbf{Fun}(\mathbb{A}^1) - \text{modules}.$$

This is a sheaf via defining $M \in \mathbf{QCoh}(\mathbb{A}^1), \in \mathbb{A}^1 \mapsto M_a \coloneqq M/(t-a)M \in \mathbf{Vect}$. E.g., $\mathscr{O}_{\mathbb{A}^1} = \mathbb{C}[t]$ and $(\mathscr{O}_{\mathbb{A}^1})_a \cong \mathbb{C}$. This is "constant".

Let $\mathscr{O}_{\mathbb{A}^1} \cong T_{\mathbb{A}^1} = \mathbf{Vect}(\mathbb{A}^1) = \mathbb{C}[t]\frac{d}{dt} \cong \mathbb{C}[t]$ with the last iso via $f\frac{d}{dt} \mapsto f$.

Question 4.1. What about $\mathbb{P}^1$? Here $\mathbf{Fun}(\mathbb{P}^1) \cong \mathbb{C}$.

Similarly, $\mathbf{QCoh}(\mathbb{P}^1)$ should contain vector bundles on $\mathbb{P}^1$, etc.

But the above definition of **QCoh** is only correct for affine varieties, i.e. those of the form

$$\{f_1 = \cdots = f_m = 0\} \subseteq \mathbb{A}^n$$

for some $n \in \mathbb{N}, f_1, \cdots, f_m \in \mathbb{C}[t_1, \ldots, t_n]$.

This isn't global, for affine X , $\dim \mathbf{Fun}(X) < \infty \implies X$ is finite. Affine sets have infinite dimensional global functions. Thus $\mathbb{P}^1$ isn't affine.

Example 4.8. Consider $\mathbb{A}^1 \setminus \{0\}$. This is relevant as the overlap between the two charts on $\mathbb{P}^1$. This doesn't look affine, but it is affine via $\mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^2$ via $t \mapsto (t, t^{-1})$. So $\mathbb{A}^1 \setminus \{0\} \xrightarrow{\sim} \{xy = 1\}$.

This is nice because the overlap between the charts it then affine. So $\mathbf{Fun}(\mathbb{A}^1 \setminus \{0\}) = \mathbb{C}[t, t^{-1}]$.

Finally, consider $j : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$. If we have $M \in \mathbf{QCoh}(\mathbb{A}^1)$, we get $j^*M = M|_{\mathbb{A}^1\setminus\{0\}} \coloneqq \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t]} M \in \mathbf{QCoh}(\mathbb{A}^1 \setminus \{0\})$. This notation is unambiguous because j is an open embedding.

Now on $\mathbb{P}^1$, we have the coordinates $s = t^{-1}$. In particular, $\mathbb{C}[t,t^{-1}] = \mathbb{C}[s,s^{-1}]$. So we can describe $\mathbf{QCoh}(\mathbb{P}^1) \coloneqq \{(M_0, M_\infty, \alpha) | M_0 \in \mathbf{QCoh}(\mathbb{A}^1_0), M_\infty \in \mathbf{QCoh}(\mathbb{A}^1_\infty), \alpha : M_0|_{\mathbb{A}^1_0\setminus\{0\}} \cong M_\infty|_{\mathbb{A}^1_\infty\setminus\{\infty\}}\}$ (here the M_0 is different from the M_0 from before). Because the restriction sets are affine, this doesn't cause any definitional issues as restriction sheaves.

E.g., $\mathscr{O}_{\mathbb{P}^1} = (\mathscr{O}_{\mathbb{A}^1_0}, \mathbb{A}_\infty, \mathrm{id}_{\mathscr{O}})$. We also have the tangent sheaf $\mathscr{T}_{\mathbb{P}^1} = (\mathscr{T}_{\mathbb{A}^1_0}, \mathscr{T}_{\mathbb{A}^1_\infty}, \mathrm{id}_{\mathscr{T}})$. We have that $\mathscr{T}_{\mathbb{A}^1_0\setminus\{0\}} = \mathbb{C}[t,t^{-1}]\frac{d}{dt} = \mathbb{C}[s,s^{-1}](-s^2)\frac{d}{ds} = \mathbb{C}[s,s^{-1}]\frac{d}{ds} = \mathscr{T}_{\mathbb{A}^1_\infty\setminus\{\infty\}}$.

Let $\rho : \mathbb{P}^1 \to pt$ leads to $p_* : \mathbf{QCoh}(\mathbb{P}^1) \to \mathbf{Vect}$ which is the "global sections"

$$\rho_*\mathscr{F} = \Gamma(\mathbb{P}^1, \mathscr{F}).$$

If $\mathscr{F} = (M_0, M_\infty, \alpha)$, then

$$\Gamma(\mathbb{P}^1, \mathscr{F}) = \ker(M_0 \oplus M_\infty \to M_\infty|_{\mathbb{A}^1_\infty\setminus\{\infty\}})$$

with the latter map sending $(m_0, m_\infty) \mapsto \alpha(m_0) - m_\infty$, with the latter term helping deal with overlap.

For example, $\Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) = \mathbf{Fun}(\mathbb{P}^1) \cong \mathbb{C}$ and $\Gamma(\mathbb{P}^1, \mathscr{T}_{\mathbb{P}^1}) = \mathbf{Vect}(\mathbb{P}^1) \cong \mathfrak{sl}_2$.

One could imagine instead of id, we have a twist. This twist gives us the standard structure sheaf. We have an action of $\mathbf{SL}_2$ on this via action on the structure sheaf, so a natural question is what finite dimensional representations come from this?

4.3. D-modules.

Let $D_{\mathbb{A}^1} \coloneqq \mathbf{Diff}(\mathbb{A}^1)$.

Definition 4.9. Define $\boldsymbol{D} - mod(\mathbb{A}^1)$ as the $D_{\mathbb{A}^1}$ modules.

E.g., we have $\mathscr{D}_{\mathbb{A}^1} \coloneqq D_{\mathbb{A}^1}$ thought of as a left module over itself. Another example is $\mathscr{O}_{\mathbb{A}^1} \coloneqq \mathbb{C}[t]$, but the action here isn't as clear. The action is the obvious action: $\frac{d}{dt} \cdot f = f'$. This is what we want to think of as the constant sheaf.

Now we begin handwaving.

Similarly, we can think about $\boldsymbol{D} - mod(\mathbb{A}^1 \setminus \{0\})$. Here the result is that

$$D_{\mathbb{A}^1\setminus\{0\}} = \oplus_{n\geq 0}\mathbb{C}[t,t^{-1}]\frac{d^n}{dt^n}.$$

We have the function $j : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$. If we have $M \in \boldsymbol{D} - mod(\mathbb{A}^1)$, we produce $j^*M = \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t]} M$. The action of $D_{\mathbb{A}^1\setminus\{0\}}$ is given by

$$\xi \in \mathbf{Vect}(\mathbb{A}^1 \setminus \{0\}), m \in M$$

$$\xi \cdot m \in j^*M, f\ in \mathbb{C}[t,t^{-1}]$$

$$\xi(f \otimes m) = (\xi f) \otimes m + f(\xi \cdot m)$$

this way it satisfies a Lie-algebra looking rule.

Analogously for $\mathbf{QCoh}(\mathbb{P}^1)$, define

$$\boldsymbol{D} - mod(\mathbb{P}^1) \coloneqq \{(M_0, M_\infty, \alpha) | \cdots\}$$

We have

$$\mathrm{oblv} : \boldsymbol{D} - mod(\mathbb{P}^1) \to \mathbf{QCoh}(\mathbb{A}^1)$$

which is the forgetful functor forgetting the action. Then define

$$\Gamma(\mathbb{P}^1, M) \coloneqq \Gamma(\mathbb{P}^1, \mathrm{oblv}M).$$

Note that this isn't the direct image in $\boldsymbol{D}-mod$. By the naturality of this construction, we have $D_{\mathbb{P}^1} \circlearrowright \Gamma(\mathbb{P}^1, \mathrm{oblv} M) = \Gamma(\mathbb{P}^1, \mathscr{D}_{\mathbb{P}^1})$. So $\Gamma : \boldsymbol{D}-mod(\mathbb{P}^1) \to D_{\mathbb{P}^1}-mod$.

Theorem 4.10. *This is an equivalence!*

This is summarized as $\mathbb{P}^1$ being D-affine.

Last time, Beilinson-Bernstein gave us that

$$D_{\mathbb{P}^1} \cong U(\mathfrak{sl}_2)_0.$$

Similarly, here $\Gamma : \boldsymbol{D} - mod(\mathbb{P}^1) \xrightarrow{\sim} U(\mathfrak{sl}_2)_0 - mod$ called $\boldsymbol{Loc}$.

Now we want to localize $\mathscr{O}_{[0]}$.

Problem: this doesn't quite fit, as $U(\mathfrak{sl}_2)_0$ has the Casimir acting as 0 whereas $\mathscr{O}_{[0]}$ acts diagonally.

So $\mathscr{O}_{[0]} \not\subset U(\mathfrak{sl}_2)_0 - mod$. But we can tweak $\boldsymbol{Loc}$ to get $\boldsymbol{Loc}' : \mathscr{O}_{[0]} \hookrightarrow \boldsymbol{D} - mod(\mathbb{P}^1)$. This isn't an equivalence, but is fully faithful.

Let $i : \{\infty\} \hookrightarrow \mathbb{P}^1$. Then $i_{dR,*} : \mathbf{Vect} = \boldsymbol{D} - mod(pt) \rightleftarrows \boldsymbol{D} - mod(\mathbb{P}^1) : \mathrm{id}^!_R$. This notation of functions and arrows on each side is adjointness.

Finally, $\boldsymbol{Loc}'$ has image

$$\boldsymbol{D} - mod(\mathbb{P}^1)^{\mathcal{G}_a}_{hol} := \{M \in \boldsymbol{D} - mod(\mathbb{P}^1) | j^* M \cong \mathscr{O}^{\oplus r}_{\mathbb{A}^1}\}$$

(notation may be slightly off) and $\dim(Ri^!_{dR} M) < \infty$.

Let $\delta_\infty := i_{dR,*}\mathbb{C}$. This tweaked equivalence $\boldsymbol{Loc}' : \mathscr{O}_{[0]} \xrightarrow{\sim} \boldsymbol{D} - mod(\mathbb{P}^1)^{\mathcal{G}_a}_{hol}$ that sends $\mathrm{triv}(\mathbb{C}) = L^0 \mapsto \mathscr{O}_{\mathbb{P}^1}$, $L^{-2} \mapsto \delta_\infty$, $M^0 \mapsto j_! \mathscr{O}_{\mathbb{A}^1}, (M^0)^* \mapsto j_* \mathscr{O}_{\mathbb{A}^1}$, and finally $P^{-2} \mapsto \mathscr{P}_\infty$. I.e., these objects in the image are the simple modules in the latter category. This last object is still mysterious. We are still on the hunt for a geometric description of these representations.

We have the exact sequence

$$0 \to j_! \mathscr{O}_{\mathbb{A}^1} \to \mathscr{P}_\infty \to \delta_\infty \to 0.$$

The computation $\mathrm{Ext}^1_{\boldsymbol{D}-mod(\mathbb{P}^1)}(\delta_\infty, j_! \mathscr{O}_{\mathbb{A}^1}) = \mathrm{Ext}^1(\mathbb{C}, Ri^!_{dR,j}; \mathscr{O}_{\mathbb{A}^1})$. The latter equals $H^1(R^1 i_{dR} j_! \mathscr{O}_{\mathbb{A}^1})$. By the contradiction principle, this is isomorphic to $H^{dR}_0(\mathbb{A}^1 \setminus \{0\})[-1]$. We also have the isomorphism to $H^{dR}_0(\mathbb{A}^1 \setminus \{0\}) \cong \mathbb{C}$.

R is the right derived functor.