REPRESENTATION THEORY

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Contents

1. 6/13 - Justin Campbell

1.1. **Symmetry Groups.** Representations come in many flavours, and the most important ones are representations of groups—a mathematical object that captures the symmetry of some structure. A bit more formally is that we want some way of acting on itself that is invertible, i.e. "automorphisms". We would also want them to be able to put together via composition.

Formally, a group is

Definition 1.1. A set G with a function $G \times G \to G$ that is our multiplication operation s.t.

(1) there is an identity element e such that $(e, g) \mapsto e$ for all elements $g \in G$
(2) associativity, i.e. $a(bc) = (ab)c$
(3) inverse, i.e. $\forall g \in G$, $\exists g^{-1}$ s.t. $(g, g^{-1}) \mapsto e$.

Example 1.2. $X =$ a square. Then we have the symmetry of a square being the set of 4 rotations and 4 reflections and the operation of doing them one after the other. This is called D_8, the dihedral group of order 8 (order of a group is the group size).

Let $r =$ the rotation through $\frac{\pi}{2}$ counter clock wise and s the reflection over the x-axis. Knowing these two operations allow us to generate D_8: $D_8 = \{e, r, r^2, r^3, rs, r^2s, r^3s\}$. A word of caution: we read the actions right to left, like functions.

We have some relations with r, s, namely $r^4 = e, s^2 = e, sr = r^3s$.

Example 1.3. Let $X = \boldsymbol{C}^n$, the n-dimensional complex vector space. It turns out that $\{$invertible linear transformations $\boldsymbol{C}^n \to \boldsymbol{C}^n\} \cong \mathrm{GL}_n(\boldsymbol{C})$ ($n \times n$ matrices with coefficients in $\boldsymbol{C}$) with the isomorphism being one that respects the extra structure of composition and multiplication, i.e. composing linear transformations on the left correspond to multiplication of the matrices on the right.

Easiest groups to understand are commutative groups, i.e. abelian groups. Generally though, matrices aren't commutative. But with $n = 1$, $\mathrm{GL}_1(\boldsymbol{C}) = \boldsymbol{C}^\times = (\boldsymbol{C} \setminus \{0\}, \cdot)$ in which it is commutative.

In a general group, $x, y \in G$ are conjugate if $\exists g \in G$ s.t. $gxg^{-1} = y$. We can fairly easily see that conjugacy is an equivalence relation. This captures some idea of the failure of G to be abelian. For example, if G is abelian, then $gxg^{-1} = x$, so all conjugates are trivial. So the only conjugacy class of $x \in G$ is $\{x\}$.

Definition 1.4. The set of x s.t. $\forall g \in G, xg = gx$ (which is the same as the conjugacy class of x is $\{x\}$) is the center. Denote this set as $Z(G)$.

Example 1.5 (The conjugacy classes in D_8)**.** Basic fact: If G is finite and $C \subseteq G$ is a conjugacy class, then $\#C | \#G$. As r^2 is in the center and r^3 isn't, $Z(D_8) = \{e, r^2\}$.

The remaining conjugacy classes are $\{r, r^3\}, \{s, r^2 s\}, \{rs, r^3 s\}$. This can be summarized by the class equation of D_8: $8 = \sum \#C$ (sum of sizes of all conjugacy classes) as $8 = 1 + 1 + 2 + 2 + 2$.

Theorem 1.6. *Two matrices in $GL_n(\boldsymbol{C})$ are conjugate iff they have the same Jordan normal form.*

Example 1.7. Any matrix in $\mathrm{GL}_2(\boldsymbol{C})$ is conjugate to one of the following:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

They are called scalar/central, regular semisimple, and non-semisimple matrices respectively.

The only redundancy in the above list is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}.$$

A is conjugate to central or regular semisimple iff $\boldsymbol{C}^2$ has a basis consisting of eigenvectors for A.

A is conjugate to (ii) iff A has two distinct eigenvalues.

1.2. **Representations of Finite Groups.** What is a representation? The goal is to represent algebraic structures as matrices, passing algebra to linear algebra.

Definition 1.8. An n-dimension *representation* of G is a map

$$\pi : G \to \mathrm{GL}_n(\boldsymbol{C})$$

satisfying

$$\pi(xy) = \pi(x)\pi(y).$$

I.e. π is a group homomorphism, and in general homomorphism refers to a function that respectives the algebraic object's structure, which here is the group multiplication.

Example 1.9. $\pi : D_8 \to \mathrm{GL}_2(\boldsymbol{C})$. Once we determine where to send r, s, then we have determined all of D_8 since they generate it. Then $\pi(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \pi(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Exercise 1.2.1. This is a well-defined homomorphism, i.e. it also has the needed relations of r, s.

Definition 1.10. Two n-dimensional vector spaces representations π_1, π_2 are called isomorphic (or conjugate) if there is a map $A \in \mathrm{GL}_n(\boldsymbol{C})$ s.t.

$$A^{-1}\pi_1(g)A = \pi_2(g)\forall g \in G.$$

A well-studied problem is to classify all representations of a fixed group G up to isomorphism. A useful tool to control the size of the dimension is through the fact that we have "factorizations" of representations.

Definition 1.11. A **subrepresentation** of $\pi : G \to \mathrm{GL}_n(\boldsymbol{C})$ is a vector subspace $V \in \boldsymbol{C}^n$ s.t. π is a representation of V, i.e. $\pi(g)(v) \in V \forall v \in V, \forall g \in G$.

Note: the hard part is often showing that it is closed under $\pi(g)$.

Note 2: if we define $\mathrm{GL}_n(\boldsymbol{C})$ as the set of matrices, then we would need to specify a basis. This is where the utility of thinking of $\mathrm{GL}_n(\boldsymbol{C})$ as the set of invertible transformations comes in.z

Definition 1.12. A $n \neq 0$ dimensional representation π is **irreducible** if the only subrepresentations of π are $\{0\}$ and $\boldsymbol{C}^n$.

These form sort of our prime numbers of representations.

2. 6/18 - Representation Theory of Finite Groups Continued

We now discuss the simplest way to put together two representations into a larger one. Say we have two representations $\pi_1 : G \to \mathrm{GL}_m(\boldsymbol{C}), \pi_2 : G \to \mathrm{GL}_n(\boldsymbol{C})$. Then we have a representation $\pi_1 \oplus \pi_2$ which is $m + n$ dimensional given by

$$\pi_1 \oplus \pi_2(g) = \begin{pmatrix} \pi_1(g) & 0 \\ 0 & \pi_2(g) \end{pmatrix}.$$

A more complicated way of putting them together would be if we had some terms in the anti-diagonal instead of 0.

Theorem 2.1 (Maschke). *Any representation of a finite group of a vector space over $\boldsymbol{C}$ is isomorphic to a direct sum of irreducible representations.*

Remark 2.2. The decomposition above are uniquely determined (including multiplicities) up to isomorphism.

Example 2.3. $\pi : D_8 \to \mathrm{GL}_2(\boldsymbol{C})$ via mapping $r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This is irreducible.

To show this, it suffices to show that there is no 1-dimensional subrepresentation in $\boldsymbol{C}^2$. The eigenspaces for $\pi(s)$ is $\boldsymbol{C} \cdot (1, 0), \boldsymbol{C} \cdot (0, 1)$ but the eigenvectors of $\pi(r)$ doesn't have any eigenvectors in $\boldsymbol{R}^2$.

Theorem 2.4 (A). *Let G be a finite group. The number of isomorphism classes of irreducible representations equals the number of conjugacy classes in G.*

Corollary 2.5. *If G is abelian, the number of irreducible representations is the number of elements of G.*

Theorem 2.6 (B). *Say G has r irreducible representations up to isomorphism of dimension $n_1, \ldots, n_r$. Then $\#G = n_1^2 + n_2^2 + \cdots + n_r^2$.*

Corollary 2.7. *The two above theorems combine to imply that all irreducible representations of G are one dimensional. Thus in abelian groups, representations can be simultaneous diagonalized. This is like the linear algebra theorem that commuting semisimple matrices can be simultaneously diagonalized.*

Theorem 2.8 (C)**.** *If G has an irreducible n-dimensional representation, then $n|\#G$.*

Example 2.9. $\psi_1 : D_8 \to \boldsymbol{C}^\times$ by sending $g \mapsto 1$ which is called the trivial representation.

We also have

- $\psi_2 : D_8 \to \boldsymbol{C}^\times, r \mapsto 1, s \mapsto -1$
- $\psi_3 : D_8 \to \boldsymbol{C}^\times, r \mapsto -1, s \mapsto 1$
- $\psi_4 : D_8 \mapsto \boldsymbol{C}^\times, r \mapsto -1, s \mapsto -1$

Previously, we have seen that $Z(D_8) = \{e, r^2\}$. This is normal. So $G/Z(D_8) \cong \langle \overline{r} \rangle \times \langle \overline{s} \rangle$ where the bar indicates the image after reduction and the langles are the group generated by it. By the commuting relation $sr = r^3 s$, we can see that this is the Klein four group.

Hence the matrices that send $G/Z(D_8)$ to $\boldsymbol{C}^\times$ are just ψ_i.

We also have the commutator, $[G, G] = \{\text{subgroup generated by } xyx^{-1}y^{-1}, x, y \in G\}$, which is normal. The abelianization of G is $G/[G, G]$. This is the biggest possible abelian quotient.

By coincidence, $[D_8, D_8] = Z(D_8)$, so $D^{ab} = D_8/Z(D_8)$. This tells us how to write the 1-dimensional representations because the one-dimensional representations of G biject to 1-dimensional representations of its abelianization. Hence $\#\{\text{1-dimensional representations of } G\} = \#G^{ab}$.

Further, this shows that $\pi : D_8 \to \mathrm{GL}_2(\boldsymbol{C})$ is irreducible. D_8 has 5 conjugacy classes, so by Theorem A, any irreducible D_8 representations is isomorphic to $\psi_1, \psi_2, \psi_3, \psi_4, \pi$.

We can verify that Theorem B,C hold: $8 = 1 + 1 + 1 + 4$ and $1|8$, $2|8$.

Exercise 2.0.1. Do the classification problem for S_3, the symmetric group on 3 letters. Also do the quaternions.

2.1. **Character Theory.** Characters provide a powerful tool to study representations. They allow us to reformulate representations as functions and also gives us invariants of a group.

Recall that the trace of an $n \times n$ matrix $A = (a_{ij})$ is $\mathrm{tr}(A) = \sum a_{ii}$. A crucial property of the trace is that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. It follows that if B is invertible, then $\mathrm{tr}(BAB^{-1}) = \mathrm{tr}(A)$.

Definition 2.10. A **character** of a representation $\pi : G \to \mathrm{GL}_n(\boldsymbol{C})$ is a function on $\chi_\pi : G \to \boldsymbol{C}$ given by

$$\chi_\pi(g) \coloneqq \mathrm{tr}(\pi(g))$$

.

We have

$$\chi_\pi(hgh^{-1}) = \chi_\pi(g) \forall g, h \in G.$$

Definition 2.11. Hence $\chi_\pi \in \mathrm{Cl}(G)$, the set of functions invariant under conjugation, i.e. $f(hgh^{-1}) = f(g) \forall g, h \in G$.

Theorem 2.12 (A'). *For G finite,*

$$\{\chi_\pi | \pi \text{ is an irreducible representation of } G\}$$

form a basis of $\mathrm{Cl}(G)$.

Corollary 2.13. *If two representations $\chi_{\pi_1} = \chi_{\pi 2}$ have the same character (as functions), $\pi_1 \cong \pi_2$.*

Corollary 2.14. *Theorem A is a corollary of Theorem A'! For any conjugacy class $C \subseteq G$, define $f_C(g) := 1$ if $g \in C$, 0 otherwise. The set $\{f_C(g) | C \subseteq G \text{ is a conjugacy class}\}$ form a basis of $\mathrm{Cl}(G)$. Hence by vector space dimension theory, the size of the above set which is the number of conjugacy classes equals the number of bases in Theorem A', which is the number of isomorphic irreducible representations.*

Definition 2.15. The **character table** of G lists the values of χ_π for all irreducible representations π.

Example 2.16. Character Table of D_8.

	e	r	r^2	s	rs
ψ_1	1	1	1	1	1
ψ_2	1	1	1	-1	-1
ψ_3	1	-1	1	1	-1
ψ_4	1	-1	1	-1	1
π	2	0	-2	0	0

Unfortunately, subgroup and normal subgroup structure are subtle things that representation theory doesn't capture super well.

Example 2.17. Character Table of A_5, the permutation group on A_5 that are even. We have that $\#A_5 = 60$ and it is the smallest nonabelian simple group.

	e	(12)(34)	(123)	(12345)	(12354)
	1	1	1	1	1
	4	0	1	-1	-1
	5	1	-1	0	0
	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

A simple group has a trivial abelianization, so there is only one 1-dimensional representation.

The first two rows come from reducing the regular representation of A_5 (there is a natural representation of A_5 on $\boldsymbol{C}^5$), namely the four dimensional is from representing the space with coordinates summing to 0. It turns out that these first two rows give us the rest of the table.