TOPOLOGICAL QUANTUM FIELD THEORY (TQFTS)

VINCENT TRAN

ABSTRACT. Historically, there has been a strong connection between geometry, topology, and physics. Topology provides a good framework to formalize certain quantum phenomena mathematically. In these lectures, I will focus on the mathematical side of this story and introduce 2-dimensional Topological Quantum Field Theories (TQFTs). The main ingredients of these lectures will be manifolds, cobordisms, and some category theory. I will attempt to make the lectures accessible, and students of all levels are welcome. A good introductory reference for this topic is the book by Kock "Frobenius algebras and 2D Topological Quantum Field Theories" (`https://math.mit.edu/~hrm/palestine/koch-frobenius-algebras.pdf`)

CONTENTS

1. LECTURE 1

Definition 1.1. An n dimensional **manifold** is a topological space that locally looks like Euclidean space of dimension n.

Example 1.2. A circle, S^1, is a one dimensional manifold. And if we zoom into a point on the circle, we have an open arc, which is homeomorphic to $\mathbb{R}$.

Another is the 2-sphere, S^2.

A torus is too, T^1.

What about the figure eight? It isn't a manifold because at the crossing point, it always looks like a cross no matter how much we zoom in. Topologically, we can see that they aren't homeomorphic because all neighborhoods of the crossing point can be disconnected into 4 connected pieces while this doesn't happen in $\mathbb{R}$ (it splits into 2).

Two disjoint circles are also a manifold, so notably manifolds don't have to be connected.

Definition 1.3. A **manifold with boundary** is a n-manifold that looks like n-dimensional Euclidean space at any interior point and the upper half Euclidean space at points on the boundary.

Example 1.4. A closed line segment is a manifold with boundary.
Another is a filled in cup, i.e. beanie.

Here's some jargon: a closed manifold is a manifold without boundary.

1.1. **Orientation and in and out boundaries.**
An orientation consists of two pieces of data:

manifold	example
Σ is a closed manifold of dimension $n-1$	imagine a circle
M is a manifold with boundary of dimension n	imagine a cylinder with one end Σ

Notice that the dimension of the boundary being one less than the manifold makes sense with the examples above.

The derivative at a point of a curve is a line tangent to a curve at a point. Now choose a point $x \in \Sigma$. Next choose a basis for the tangent space at x, call it $v_1, \ldots, v_{n-1}$. Call this the positive basis. This alongside an extra vector, due to the manifold being a dimension one larger, generates the tangent space at x, $T_x M$. Call this vector ω the positive normal vector.

Define an oriented manifold to be such that, we can move the boundary Σ through the interior of the manifold and reach the other end while keeping the positive normal vector consistently oriented. Finally, we can define the in boundary to be the boundary with the positive normal vector pointing inside the manifold and the out boundary to not be.

Now we move to a very important alg top concept:

1.2. **Cobordism.**

Definition 1.5 (Oriented Cobordism)**.** Let Σ_0 and Σ_1 be oriented $(n-1)$ dimensional manifolds. A oriented cobordism from Σ_0 to Σ_1 is an oriented n-dimensional manifold whose boundary is $\Sigma_0 \sqcup \Sigma_1$.

Example 1.6. Say we have $\Sigma_0 = S^1 \sqcup S^1$ and $\Sigma_1 = S^1$. This cobordism is called the pair of pants because it looks like pants.

A similar picture happens with $\Sigma_0 = S^1 \sqcup S^1 \sqcup S^1$. This is called the alien pair of pants.

Question 1.1. For food for thought, we can realize that cobordism creates an equivalence relation on the set of n-dimensional manifolds. This is because it is obviously symmetric, transitive (just connect the manifolds), and is reflexive by just crossing it with an interval (a cylinder with the same manifold twice at the boundaries). Further, cobordisms have a category. Do cobordisms form a group?

2. Lecture 2 - Category Theory

Definition 2.1. A **category** consists of two parts of information:

- Objects
- Morphisms, i.e. arrows, between the objects

with sensible compatibility, i.e.

- all objects have an identity morphism, i.e. an arrow mapping it to itself
- composition of morphisms is a morphism

Example 2.2. We have objects $a, b, c,$ f_1, f_2, and the necessary arrows

$$\begin{tikzcd} a \arrow[loop above, "\mathrm{id}"] \arrow[r, "f_1"] \arrow[rr, bend right, "f_2 \circ f_1"] & b \arrow[loop above, "\mathrm{id}"] \arrow[r, "f_2"] & c \arrow[loop above, "\mathrm{id}"] \end{tikzcd}$$

Example 2.3. Commonly used categories are the category of groups, abelian groups (i.e. modules), vector spaces, and rings.

Example 2.4. A group is a category with one object and the elements in the group being the morphisms of the category.

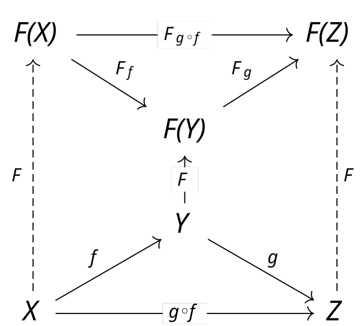
Definition 2.5. Cobordism Category: Denote the n-dimensional cobordisms by **ncob**. The objects are $(n-1)$ dimensional oriented, compact, closed manifolds. The morphisms are n-dimensional cobordisms.

We want to specifically work with 2 dimensional cobordism category. The objects are $S^1, S^1 \sqcup S^1$, etc. Intervals are eliminated because we have closed and compact. The morphisms are pants like cobordism connecting every object. This gives us a geometric category.

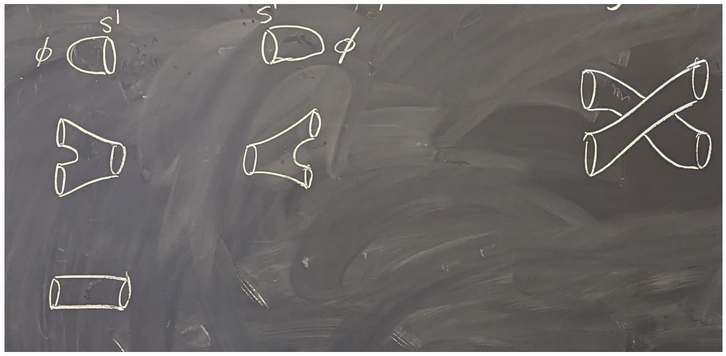
Example 2.6. The category of vector spaces over a field, **vect** . The objects are vector spaces. The morphisms are linear maps. This category is an algebraic category.

In math we love to study relations, so look for a relation between categories. This is a

Definition 2.7. functor. It is a function between categories that map objects of a category $\mathfrak{A}$ to objects of another category $\mathfrak{B}$ and morphisms in $\mathfrak{A}$ to morphisms in $\mathfrak{B}$ such that the relation the morphisms hold in $\mathfrak{A}$ hold in $\mathfrak{B}$. I.e., the diagram here:



We bring this up because there is a functor between $2-cob$ and $\mathrm{vect}_{\mathbb{R}}$. Mathematically, a TQFT is an example of a functor **ncob** $\to \mathrm{vect}_{\mathbb{R}}$. To understand this, we examine the two dimensional case. We discuss $2-cob$ because the classification of 2-manifolds is complete. Genus classifies them. This contrasts with 3 dimensional, which is very complicated. The four dimensional is impossible in a sense.

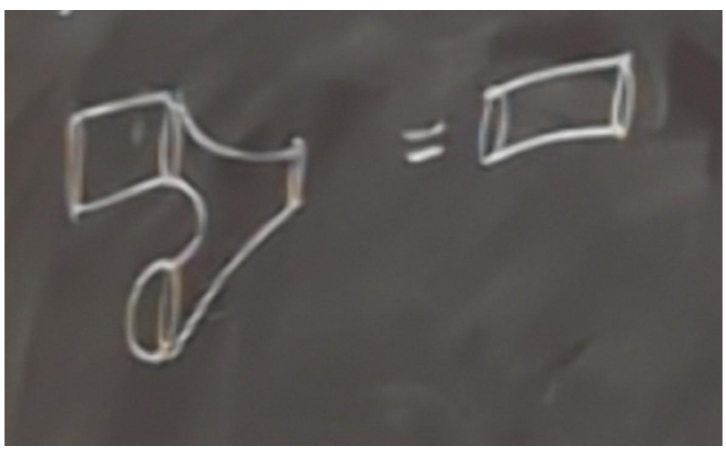


Proposition 2.8. 2−*cob is generated by the following cobordisms: a cup (cobordism between* $\emptyset$ *and* S^1*, a cobordism between* S^1 *and* $S^1 \sqcup S^1$*, a cylinder, the reflections of the first two, and a twisting connection between* $S^1 \sqcup S^1$ *and itself.*

The last one is different from two identities because as functions they map $(1,2) \mapsto (2,1)$ where $1, 2$ are the labelled copies while two identities is $(1,2) \mapsto (1,2)$. Another way to see this is that every manifold has a basepoint, sitting in one connected component. But the twist has the basepoint in a different connected component.

2.1. **Glueing Building Blocks.** If we glue together two cylinders, we see that it is another cylinder. Similarly for the other generators, the cylinder is the identity on all the generators.

What if we glue a cylinder and a cup, to the two->one? I.e. It becomes the



identity.

A special one of note is the Frobenius relation.