RIEMANN ROCH THEOREM ON GRAPHS

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Abstract. I will explain the Riemann Roch theorem, a very important theorem connecting geometry and algebra in a more hand wavy way, and then explain how it appears for graphs through the chip firing game. This will be elementary, open to anyone.

Contents

1. Riemann Roch Explained

In algebraic geometry, one often wants to study the algebra of sufficiently nice functions (often polynomials) in order to understand the geometry. A classic example is to detect multiplicities: the set of complex roots to $x^2 = 0$ and $x = 0$ (on a line) geometrically "look" the same yet algebraically they can be distinguished. Namely, the former has its functions on it being polynomials in $\mathbb{C}[x]/x^2$ while the latter has its functions being polynomials in $\mathbb{C}[x]/x$. The $x^2 = 0$ is sort of "fatter" than $x = 0$: its algebra contains the polynomial x, which it analogous to an infinitesimal around 0.

So one might naturally want to try and control the algebra of polynomials on a space with a certain set of roots and poles of a certain multiplicity, such as by trying to bound the dimension. To capture this requirement of a certain set of roots and poles of a certain multiplicity, let

$$D = \sum a_n z_n$$

where a_n is the multiplicity (positive being multiplicity of root, negative being multiplicity of a pole) at z_n. Let $\ell(D)$ be the dimension of rational polynomials satisfying at least D (i.e. poles are no worse) and $\deg D$ to be the sum of a_n. Now we can state a bound

Theorem 1.1 (Riemann-Roch Inequality)**.** *On a non-singular complex curve of genus g:*

$$\ell(D) \geq \deg D - g + 1.$$

There is also a version of this for compact Riemann surfaces, which is essentially the same as above except with meromorphic functions instead of rational. It turns out that there is a similar result to the above, but for graphs.

2. Riemann-Roch on Graphs

Let G be a graph and suppose it has weights $f \in \mathcal{M}(G)$. Let $V(G)$ be the set of vertices of a graph G and E_v the set of vertices adjacent to $v \in V(G)$. Then consider the following one player game:

(1) Goal: get all weights to be non-negative
(2) Moves: Fix a vertex.
   (a) Borrow: subtract one from each neighboring vertex and add that amount to the vertex
   (b) Loan: add one to each neighbor, subtract that amount from the vertex

In this way, the total amount of "money" in the game stays constant. For example with the top vertex doing a borrow:

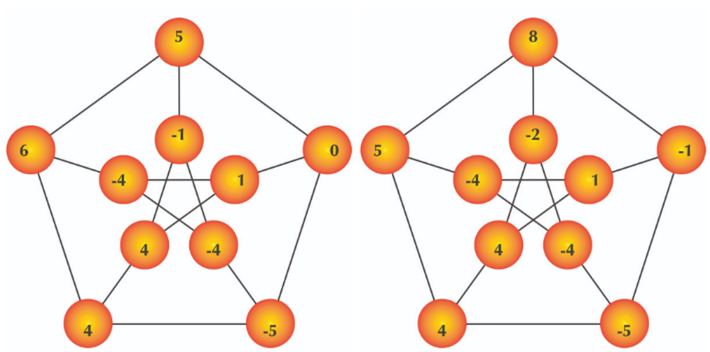

Figure 1. Taken from [2]

The moves in this game are sort of like flows, and so naturally the Laplacian is useful here. Like the Laplacian in finite heat flow, we have that for $f \in \mathcal{M}(G)$,

$$\Delta(f) = \sum_{v \in V(G)} \sum_{w \in E_v} (f(v) - f(w)).$$

There are many analogues of terms from algebraic geometry for graphs. For those who know formally about the standard Riemann-Roch, we have

AlgGeo	Graphs
Divisors	Formal Sum of Vertices (i.e. $\sum a_n v_n$ with a_n integers, v_n vertex)
Degree	$\sum a_n$
Effective Divisor	$a_n \geq 0$ for all n
Partial Ordering on Divisors	$D \geq D'$ iff $a_n \geq a'_n$ for all n
Section of Structure Sheaf	$\mathcal{M}(G) := \mathrm{Hom}(V(G), \mathbb{Z})$ (functions on vertices)
$\mathrm{Prin}(G)$	$\Delta(\mathcal{M}(G))$
Linear Equivalence $D \sim D'$	$D - D' \in \mathrm{Prin}(G)$
$\mathrm{Div}^0(G)$	Divisors D such that $\deg D = 0$
$\mathrm{Jac}(G)$	$\mathrm{Div}^0(G) / \mathrm{Prin}(G)$
Linear System $\lvert D \rvert$	Set of effective divisors $D' \sim D$

For those who don't, take the above as definitions for the graph versions.

The terminology above helps us in that it turns out that two weights on a graph can be reached by moves in the game if and only if they are linearly equivalent as divisors. Then the winning state in this language is just that D is linearly equivalent to an effective divisor.

The analogue of dimension in this context will be as follows:

Definition 2.1. Let $r(D)$ for a divisor D be -1 if $|D| = \emptyset$ and the largest $s \geq 0$ such that $|D - E_s| \neq \emptyset$ for all effective divisors E_s of degree s.

Warning 2.2. This r differs a bit from ℓ, and is merely an analogue as evident from the definition. For details see Baker's paper.

Theorem 2.3 (Riemann-Roch Inequality on Graphs)**.** *Let $g = |E(G)| - |V(G)| + 1$. For any divisor D on a graph G,*

$$r(D) \geq \deg D + 1 - g.$$

For those of y'all wondering if the inequality is strict, we have

Theorem 2.4 (Riemann-Roch Equality)**.** *Under the same setting of the Riemann-Roch Inequality,*

$$\ell(D) - \ell(K - D) = \deg D - g + 1$$

with K being the canonical divisor (too complicated to develop here)

with the analogue being

Theorem 2.5 (Riemann-Roch Equality on Graphs)**.** *Under the same setting of the Riemann-Roch Inequality on Graphs,*

$$r(D) - r(K - D) = \deg D - g + 1.$$

Here, on graphs, this mysterious K is more concrete.

Definition 2.6. The **canonical divisor of a graph** is

$$K := \sum_{v \in V(G)} (\deg(v) - 2)(v)$$

with $\deg(v)$ being the number of neighbors of v.

3. Other Analogues

Another cool analogue is that with the bilinear form

$$\langle f, D \rangle := \sum_{v \in V(G)} f(v) D(v),$$

the Lapacian of a graph is self-adjoint. This is a graph theoretic analogue of Weil reciprocity theorem on a Riemann surface. See Baker's paper for more details.

One can also prove the Kirchhoff Matrix Tree Theorem. See [2] for a reference and other results.

References

[1] Matthew Baker and Serguei Norine. "Riemann-Roch and Abel-Jacobi theory on a finite graph". In: (2007). arXiv: math/0608360 [math.CO]. URL: https://arxiv.org/abs/math/0608360.

[2] *Riemann-Roch for Graphs and Applications*. [Online; accessed 2. Aug. 2024]. Feb. 2014. URL: https://mattbaker.blog/2013/10/18/riemann-roch-for-graphs-and-applications.