

THE ENIGMATIC $\mathbf{Z}/12\mathbf{Z}$

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ABSTRACT. This paper will introduce a new conception of a topology that abstracts enough from a normal topology to allow for cohomology theories. We then define what a family of elliptic curves is, and then define the families of elliptic curves of interest, the modular families. Finally, we will then formally construct the moduli space of (modular) families of elliptic curves, from where we will compute the Picard group of the moduli space of families of elliptic curves (and find that it is isomorphic to $\mathbf{Z}/12\mathbf{Z}$!).

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1. A NEW CONCEPTION OF TOPOLOGY

The Weil conjectures state that the number of solutions to a system of polynomials¹ over $\mathbf{F}_{p^n}$ is a sum of geometric series in n . Grothendieck noticed that the formulas predicted by the Weil conjectures could be proven with a good cohomology theory on varieties, namely via the Lefschetz fixed point theorem (since $\mathbf{F}_{p^n}$ points are fixed points of the $\text{Gal}(\overline{\mathbf{F}_p} : \mathbf{F}_{p^n})$ action). But the classic topologies we knew of on varieties (such as the Zariski, fpqc, flat, etc.) failed to capture the “absolute” cohomology theory given by considering the variety as a complex manifold. Famously, Serre once constructed an example in which the cohomology with the Zariski topology of two conjugate curves [Ser03] were different. But conjugate curves must have the same cohomology over $\mathbf{C}$! Hence the classical topologies are not good enough!

This motivated Grothendieck to invent a new idea of a “space” in the hopes that it can express the “absolute” cohomology theory algebraically. In particular, he wanted to generalize Čech cohomology. He saw that in order to define Čech cohomology, we need intersections and the idea of a cover to be in a topology. He then isolated three key axioms for a cover:

- (a) the identity automorphism is a cover

¹Assuming the system is decent in that the variety they form is smooth

- (b) coverings are transitive
- (c) coverings can be intersected with a fixed open set to form a cover.

As is typical in the Grothendieck style of algebraic geometry, we can try translating these into a categorical language to hope to create a “rising sea”². We can try “categorizing” a topological space X by letting the objects be open sets and morphisms be inclusions. Then intersections correspond to the category having fibred products and having covers corresponds to having a set of morphisms with a fixed target obeying three axioms:

- (a) If $\phi : U \cong V$, then $\{U \xrightarrow{\phi} V\}$ is a cover.
- (b) If $\{V_{ij} \rightarrow U_i\}$ is a cover of U_i and $\{U_i \rightarrow U\}$ is a cover of U , then $\{V_{ij} \rightarrow U\}$ is also a cover of U . This property corresponds to transitivity of covers.
- (c) If $\{U_i \rightarrow U\}$ is a cover of U , then for all open V ,

$$\{V \times_U U_i \rightarrow V\}$$

is a cover of V . This property is the analogue of intersecting with a fixed open set.

Thus

Definition 1.1. A **Grothendieck topology** is a category with fibred products and a collection of sets of morphisms with a fixed target, $\{U_\alpha \rightarrow U\}$ that has the above three properties:

The objects are then called **open sets** and the collection of sets is the **collection of covers**.

From now on we call a Grothendieck topology just a topology.

Remark 1.2. Using this language, one is able to prove a correspondence between sheaves of abelian groups and π modules when π is a group acting freely and discontinuously on a simply connected space. See [Mum65, pp. 39–41].

Just like with a space, there is a nature idea of a continuous map, via preimages:

Definition 1.3. A **continuous map** of topologies $F : T_1 \rightarrow T_2$ is a functor from the open sets of T_2 to T_1 such that

- (a) F takes final objects to final objects
- (b) F takes fibre products to fibre products
- (c) F takes covers to covers

By tradition, we denote the open set in T_1 associated to $U \in T_2$ by $F^{-1}(U)$. Critically, we have an idea of a sheaf on a topology:

Definition 1.4. A **sheaf** (with values in a category $\mathcal{C}$) $\mathfrak{F}$ on a topology T is a contravariant functor $T \rightarrow \mathcal{C}$ such that for all covers $\{U_\alpha \rightarrow U\}$, the following is an equalizer

$$\mathfrak{F}(U) \rightarrow \prod \mathfrak{F}(U_\alpha) \rightrightarrows \prod \mathfrak{F}(U_\alpha \times_U U_\beta).$$

A **morphism of sheaves** is a natural transformation of the functors between two sheaves.

²Grothendieck famously said of his problem solving philosophy, “The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it... yet it finally surrounds the resistant substance.”

As we have continuous maps and sheaves, we have pullback sheaves via $F_*\mathfrak{F}(U) := \mathfrak{F}(F^{-1}(U))$. With a theory of sheaves comes a theory of cohomology via the derived functor perspective: suppose T has a final object X (in a classical topology turned into a Grothendieck topology, this would be the entire space). Then we can define global sections via $\Gamma(\mathfrak{F}) := \mathfrak{F}(X)$, allowing us to consider injective or flabby resolutions of $\mathfrak{F}$ and applying the global section functor to them. So we have higher cohomology $H^i(T, \mathfrak{F})$.

For simplicity, henceforth all schemes are separated schemes of finite type over an algebraically closed field k .

Example 1.5 (Zariski Topology). Let X_{Zar} be the Grothendieck topology associated to the Zariski topology.

Definition 1.6. A **morphism** $f : X \rightarrow Y$ is **étale** if for all points $y \in Y$, $f^{-1}(y)$ is finite and for all $x \in f^{-1}(y)$, the map $f^* : \mathfrak{o}_{Y,y} \rightarrow \mathfrak{o}_{X,x}$ induces an isomorphism $f^* : \hat{\mathfrak{o}}_{Y,y} \rightarrow \hat{\mathfrak{o}}_{X,x}$.

Proposition 1.7. A morphism being étale is also equivalent to

- (a) f being flat
- (b) having scheme theoretic fiber $f^{-1}(y)$ being reduced finite set
- (c) $\forall x \in f^{-1}(y)$, $m_x = f^*(m_y) \cdot \mathfrak{o}_{X,x}$.

Proof. See [Mil13, Proposition 2.9]. $\square$

The intuition for étale morphisms is that it is the algebraic version of being a local homeomorphism.

Example 1.8 (Étale Topology). Define $X_{\text{ét}}$ to be the topology with open sets étale maps $U \rightarrow X$ and a morphism of $p : U \rightarrow X$ and $q : V \rightarrow X$ to be a morphism of schemes $U \rightarrow V$ over X . Then define $\{U_\alpha \xrightarrow{p_\alpha} U\}$ to be a cover if $U = \cup p_\alpha(U_\alpha)$.

Using this topology, Artin and Mazur was able to prove that there is an isomorphism $H^i(X_{\text{ét}}, \mathbf{Z}/n) \rightarrow H^i(X_{\mathbf{C}}, \mathbf{Z}/n)$ when X is nonsingular and $X_{\mathbf{C}}$ is X considered as a complex manifold [Art69]! Thus the analytic object on the right was related to the algebraic object on the left!

Here are some examples of other topologies on X :

Definition 1.9. The topology $X_{(\text{smooth}, \text{étale})}^*$ has as open sets *arbitrary* morphisms $U \rightarrow X$ and morphisms of open sets commutative diagrams

$$\begin{array}{ccc} U & \xrightarrow{\quad} & V \\ & \searrow & \swarrow \\ & X & \end{array}$$

Finally, $\{U_\alpha \xrightarrow{f_\alpha} U\}$ is a cover if $U = \cup f_\alpha(U_\alpha)$ and each f_α is (smooth, étale).

The nice thing about these topologies is that they are enlarged when compared to X_{Zar} and $X_{\text{ét}}$ yet still have restrictions on the covers giving us some structure. All of these topologies can be seen to be related by a series of continuous maps

$$X_{\text{flat}}^* \rightarrow X_{\text{smooth}}^* \rightarrow X_{\text{étale}}^* \rightarrow X_{\text{ét}} \rightarrow X_{Zar}.$$

In the next section, we will begin studying families of curves and see that they form various topologies. The case with the richest theory is that of elliptic curves,

being sophisticated objects, having genus 1, yet not too complex that it is unapproachable. Furthermore, over $\mathbf{C}$, there are multiple approaches to elliptic curves that one can hope to reconcile, giving us more to work with. We then define the moduli topology to let us understand the “universal family of elliptic curves”. The case of elliptic curves is one of the earliest instances of a need for stacky ideas, as the universal family of elliptic curves that we would want doesn’t exist as a topological space.

In section 3, we define what an invertible sheaf on this “universal family of elliptic curves” is and see that there is a cohomological interpretation of the group of invertible sheaves. Finally, in section 4 we will compute what this group actually is.

2. MODULI TOPOLOGIES

In this section we will define the topology on the set of families of elliptic curves. Most of our results will be focused on elliptic curves in particular, but for a more general treatment of genus g , see [Mum65]. Having a topology helps provide some geometric intuition to the set of families of elliptic curves. This is needed because unlike simpler cases of families of lines or circles (which are parameterized “universally” by angle and radius respectively), there is no “universal” way to parameterize families of elliptic curves.

Definition 2.1. A **family of curves** is a flat projective morphism $\pi : \mathfrak{X} \rightarrow \mathcal{S}$ such that all fibers over closed points in $\mathcal{S}$ are curves.

Call $\mathcal{S}$ the **base**.

Thus we can think of a family of curves as a set of curves C_s that vary with structure determined by the morphism (such as a continuous morphism enforcing continuity) as s varies in $\mathcal{S}$. For example, if $\mathcal{S} = \mathbf{R}^+$, then one can think of it as curves evolving in time. We care about families of curves because by enforcing a parameterization, we are able to think about how curves change with a parameter.

Definition 2.2. A **family of elliptic curves** is a family of non-singular genus 1 curves $\pi : \mathfrak{X} \rightarrow \mathcal{S}$ and a section $s : \mathcal{S} \rightarrow \mathfrak{X}$ such that $\pi \circ s = \text{id}$.

A section is then an identification of a base point on each elliptic curve. We do this because a generic curve has infinitely many automorphisms, while an elliptic curve has finitely many automorphisms (since they must fix the basepoint). These automorphisms matter because if we had a “universal” family of curves, the automorphisms would appear over the same point in the “universal” family. But we want étaleness, so fibres should be finite. This discussion will make more sense later when fine moduli spaces are discussed.

Example 2.3. Let $\mathbf{A}_j = \text{Spec } k[j]$ and $S_0 = \mathbf{A}_j \setminus \{0, 12^3\}$. Then let $\mathfrak{X}_0$ be the closed subscheme of $\mathbf{P}^2 \times S_0$ defined by the vanishing of

$$y^2z - x^3 - \frac{27}{4} \cdot \frac{12^3 - j}{j} (xz^2 + z^3).$$

It is easy to see that the j -invariant is a morphism (given coordinates, just isolate for j), hence we have $\pi_0 : \mathfrak{X}_0 \rightarrow S_0$. A computation shows that the j -invariant of

$$y^2 = x^3 + \frac{27}{4} \cdot \frac{12^3 - j_0}{j_0} (x + 1)$$

is j_0 . Thus, we have a section $\epsilon : \mathcal{S} \rightarrow \mathfrak{X}$ by picking the point at infinity via $S \rightarrow (0, 1, 0) \times S \subseteq \mathfrak{X}$.

We are in a categorical setting, so we also want to define morphisms of families:

Definition 2.4. A **morphism of a family of curves** is a diagram

$$\begin{array}{ccc} \mathfrak{X}_1 & \longrightarrow & \mathfrak{X}_2 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \mathcal{S}_1 & \longrightarrow & \mathcal{S}_2 \end{array}$$

making this a fibre product diagram.

Definition 2.5. An **(smooth/étale) morphism of families** is a morphism of families such that the morphism between the bases is (smooth/étale).

A morphism of families of elliptic curves is then just a morphism of families that preserves the basepoint, i.e. the following diagram holds for a morphism F between families of elliptic curves $\mathcal{X} \rightarrow \mathcal{S}$, $\mathcal{Y} \rightarrow \mathcal{T}$ with sections s, t respectively,

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathfrak{Y} \\ s \uparrow & & \uparrow t \\ \mathcal{S} & \xrightarrow{f} & \mathcal{T} \end{array}$$

It turns out there is an equivalent, more “homotopic” definition of a smooth morphisms making them like fibre spaces:

Proposition 2.6. *A map $X \rightarrow Y$ is smooth if for all local finite dimensional k -algebras A with an ideal I that squares to zero and maps such that*

$$\begin{array}{ccc} \mathrm{Spec} A/I & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec} A & \longrightarrow & Y \end{array}$$

induces the dotted map.

Proof. See [Stacks, Tag 02HT] □

This definition will be helpful because this definition focuses on test diagrams and morphisms, allowing us to define smoothness when the object X is formal.

Of course, to have a topology, we need fibre products. So we need to show that a fibre product of families of curves exist.

Theorem 2.7. *Let $\pi_1 : \mathfrak{X}_1 \rightarrow \mathcal{S}_1$ and $\pi_2 : \mathfrak{X}_2 \rightarrow \mathcal{S}_2$ be two families of curves with a map into $\pi_0 : \mathfrak{X}_0 \rightarrow \mathcal{S}_0$. Then $\pi_1 \times_{\pi_0} \pi_2$ exists as a family of curves.*

The proof of this will rely on the set of isomorphisms between two (particular) schemes having a scheme structure!

Proof. Let $X = \mathfrak{X}_1 \times_{\mathcal{S}_0} \mathcal{S}_2$ and $Y = \mathfrak{X}_2 \times_{\mathcal{S}_0} \mathcal{S}_1$ be $S := \mathcal{S}_1 \times_{\mathcal{S}_0} \mathcal{S}_2$ schemes. In this way, X and Y are two “naïve” constructions of the fibre product, which fails because X might not be isomorphic to Y . So we try to consider pulling X, Y back along some $T \rightarrow S$ such that there is an isomorphism.

By [FGI14, Theorem 5.23], the functor $\mathfrak{Mor}_S(X, Y)$ sending an S -scheme T to the set of morphisms $X_T \rightarrow Y_T$ over T is representable by a scheme $\mathrm{Mor}_S(X, Y)$.

Then composition of morphisms provides a natural transformation $\mathfrak{Mor}_S(X, Y) \times \mathfrak{Mor}_S(Y, X) \rightarrow \mathfrak{Mor}_S(X, X)$. Since $\mathfrak{Mor}_S$ is representable, this gives us a composition scheme morphism $\text{Mor}_S(X, Y) \times \text{Mor}_S(Y, X) \rightarrow \text{Mor}(X, X)$.

Finally, by taking the fiber over the identity morphism, we get that $I := \text{Isom}_S(X, Y)$ exists as a scheme with the universal property of characterizing an S -scheme T and an isomorphism $X_T \rightarrow Y_T$. I claim that $X_I \rightarrow I$ is the product family. First off this is well-defined since over I , $X_I \cong Y_I$. Now suppose we have $\pi_3 : \mathfrak{X}_3 \rightarrow \mathcal{S}_3$ such that the following commutes:

$$\begin{array}{ccc} \pi_3 & \xrightarrow{f} & \pi_1 \\ \downarrow g & & \downarrow \\ \pi_2 & \longrightarrow & \pi_0 \end{array}$$

Then obviously we have a map $\mathcal{S}_3 \rightarrow \mathcal{S}_1 \times_{\mathcal{S}_0} \mathcal{S}_2$ and we have isomorphisms (over $\mathcal{S}_3$) due to morphisms of families requiring them to be fibred products:

$$\begin{array}{ccccc} \mathfrak{X}_3 & & X \times_S \mathcal{S}_3 & & Y \times_S \mathcal{S}_3 \\ & \searrow & \downarrow & \swarrow & \\ & & T & & \end{array}$$

$((\mathfrak{X}_1 \times_{\mathcal{S}_0} \mathcal{S}_2) \times_{\mathcal{S}_1 \times \mathcal{S}_2} \mathcal{S}_3 \cong \mathfrak{X}_1 \times_{\mathcal{S}_0 \times \mathcal{S}_1} \mathcal{S}_3 \cong \mathfrak{X}_1 \times_{\mathcal{S}_1} \mathcal{S}_3$ since all the maps are $\mathcal{S}_0$ morphisms) Therefore we have a canonical map $\mathcal{S}_3 \rightarrow I$. Hence, this canonically gives a morphism of families $\pi_3 \rightarrow (X_I/I)$. $\square$

Definition 2.8. Let the base scheme I in the proof above be denoted $\text{Isom}(\pi_1, \pi_2)$. The product of two families $\pi_i : \mathfrak{X}_i \rightarrow \mathcal{S}_i$ will be denoted $\pi : (\mathfrak{X}_1, \mathfrak{X}_2) \rightarrow \text{Isom}(\pi_1, \pi_2)$.

Observation 2.9. We can redo this proof without the products being fibred and without π_0 to see that absolute products also exist!

Obviously, we can use [Theorem 2.7](#) to create products of elliptic curves, by creating the section as follows: $\text{Isom}(\pi_1, \pi_2)$ maps into $\mathcal{S}_1$, $\mathcal{S}_2$ and agrees on $\mathcal{S}_3$, so by universal property, we have a unique map $\text{Isom}(\pi_1, \pi_2) \rightarrow \mathcal{S}_1 \times_{\mathcal{S}_3} \mathcal{S}_2$. Then compose this map with $s \times t$ for s, t sections.

The following proposition will allow us to more concretely think about $\text{Isom}(\pi_1, \pi_2)$.

Proposition 2.10. *Given two families $\pi_1 : \mathfrak{X}_1 \rightarrow \mathcal{S}_1, \pi_2 : \mathfrak{X}_2 \rightarrow \mathcal{S}_2$. A closed point of $\text{Isom}(\pi_1, \pi_2)$ lying over a closed point $s_1 \in \mathcal{S}_1$ is the data of*

- (a) *a closed point $s_2 \in \mathcal{S}_2$ such that there is an isomorphism between $\pi_1^{-1}(s_1)$ and $\pi_2^{-1}(s_2)$*
- (b) *a choice of isomorphism*

Proof. Obviously giving the two pieces of data determines a point of $\text{Isom}(\pi_1, \pi_2)$, as pulling back along (s'_1, s'_2) is the scheme theoretic fibre, so we need an isomorphism of these two fibres, which we have. It is closed because we have by hypothesis that the pullback along an étale map is a curve.

For the opposite direction, suppose we have a closed point $p \in \text{Isom}(\pi_1, \pi_2)$. We have the following diagram:

$$\begin{array}{ccc}
 \mathfrak{X}_1 \times \mathcal{S}_2 \times_{\mathcal{S}_1 \times \mathcal{S}_2} p & \xrightarrow{\sim} & \mathcal{S}_1 \times \mathfrak{X}_2 \times_{\mathcal{S}_1 \times \mathcal{S}_2} p \\
 & \searrow & \swarrow \\
 & p & \\
 & \downarrow & \\
 \mathfrak{X}_1 \times \mathcal{S}_2 & & \mathcal{S}_1 \times \mathfrak{X}_2 \\
 & \searrow & \swarrow \\
 & \mathcal{S}_1 \times \mathcal{S}_2 &
 \end{array}$$

Notice that p is isomorphic to its image in $(s_1, s_2) \in \mathcal{S}_1 \times \mathcal{S}_2$ because $p \rightarrow \mathcal{S}_1 \times \mathcal{S}_2$ is étale and it is lying over a closed point (this information implies that the residue field of p is an extension of the residue field of its image, but the residue field of its image is k , which is algebraically closed). Thus the pullback is the scheme theoretic fibre over p . Hence the first scheme in the isomorphism is $\pi_1^{-1}(s_1) \times \{s_2\} \cong \pi_1^{-1}(s_1)$ and the second is $\{s_1\} \times \pi_2^{-1}(s_2) \cong \pi_2^{-1}(s_2)$. Thus we have such data. $\square$

It is easy to see that the other properties required of a topology are satisfied. Hence, mirroring the definition of $X_{(\text{smooth}/\text{étale})}$, we have

Definition 2.11 (Provisional Definition). Let $\mathfrak{M}_{(\text{smooth}/\text{étale})}^*$ be the topology with open sets being families of elliptic curves and morphisms of open sets being morphisms of families of elliptic curves. Then $\{U_\alpha \xrightarrow{F_\alpha} U\}$ is a covering if each F_α is (smooth/étale) and $U = \cup F_\alpha(U_\alpha)$.

Of interest is $\mathfrak{M}_{\text{étale}}^*$ since cohomology works better with étale maps. We want there to be a final object in this category, as this will allow us to define cohomology as discussed after [Definition 1.4](#).

Observation 2.12. A final object in this category would be a fine moduli space of families of elliptic curves. But to define what a final moduli space is, we first look at what a coarse one is.

Definition 2.13. A **coarse moduli space** of families of elliptic curves is a space such that every family of elliptic curves maps into it.

Example 2.14. The space $U/\mathbf{SL}_2(\mathbf{Z})$ of the upper half plane modded out by the $\mathbf{SL}_2(\mathbf{Z})$ action is a coarse moduli space for families of complex elliptic curves. See [\[Hai14\]](#) for a proof.

Example 2.15. The j -invariant from [Example 2.3](#) defines a coarse moduli space.

As can be seen from the above two examples, a coarse moduli space isn't unique. To resolve this, one can introduce the idea of a fine moduli space which demands certain universality properties.

Definition 2.16. A **fine moduli space** of families of elliptic curves is a space $M \rightarrow T$ such that every family of elliptic curves $\pi : \mathfrak{X} \rightarrow \mathcal{S}$ has a unique map $\mathcal{S} \rightarrow T$ making $\mathfrak{X}$ the pullback of M and every map into T has a pullback that is a family of elliptic curves.

But, a fine moduli space often doesn't exist, as is the case with families of elliptic curves. The reason for this is because some elliptic curves have more automorphisms than others. This idea will motivate a stack.

We care about moduli spaces because they allow us to understand all curves geometrically. For instance, we can prove properties of curves generically, consider singular curves as limit points, and understand geometrically how elliptic curves vary as the parameter varies.

If a fine moduli space existed, then the size of each fiber over a point should be the same. Namely, the size of a fiber equals the size of the automorphism group. But this is not constant for all elliptic curves, contradicting the fact that it should be constant as the morphism on the base is étale (this will be discussed again later in the form of [Proposition 2.19](#)).

Hence we begin the next definition of the moduli topology. It is clear that absolute products are fibred products over the final object (when it exists). So because absolute products are in the topology $\mathfrak{M}_{(\text{smooth}/\text{étale})}^*$, we can formally add a final object M to the topology and not immediately lose its topology-ness.

But now we need to define a cover of the final object. Here the only requirement we need to define is property (c) of a cover: if $\{U_i \rightarrow U\}$ is a cover of U , then $\{U_i \times V \rightarrow V\}$ is a cover of V . Letting $U = M$ tells us that $\pi : \mathfrak{X} \rightarrow \mathcal{S}$ being part of a covering implies that for all over open sets $\pi' : \mathfrak{X}' \rightarrow \mathcal{S}'$, $\pi \times \pi' \rightarrow \pi'$ must be part of an étale cover. Unravelling definitions, this says that we need $\text{Isom}(\pi, \pi') \rightarrow \mathcal{S}'$ to be étale. So we have that

Definition 2.17. A family of curves $\pi : \mathfrak{X} \rightarrow \mathcal{S}$ is **(smooth/étale) over M** if for all $\pi' : \mathfrak{X}' \rightarrow \mathcal{S}'$, the map $\text{Isom}(\pi, \pi') \rightarrow \mathcal{S}'$ is (smooth/étale).

So the final form definition of the moduli topology will include this final object and its covers as just defined.

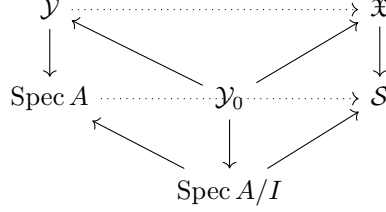
Now we try to better understand what it means for a family of curves to be smooth/étale over M . By unraveling the definition of a smooth morphism of families using [Proposition 2.6](#), we get that

Proposition 2.18. Let A be a finite-dimensional local k -algebra and I be an ideal I with diagram

$$\begin{array}{ccccc}
 & \mathcal{Y} & & & \mathfrak{X} \\
 & \downarrow & \swarrow & \searrow & \downarrow \\
 \text{Spec } A & & \mathcal{Y}_0 & & \mathcal{S} \\
 & \swarrow & \downarrow & \searrow & \\
 & & \text{Spec } A/I & &
 \end{array}$$

where $\mathcal{Y} \rightarrow \text{Spec } A$ and $\mathcal{Y}_0 \rightarrow \text{Spec } A/I$ are two families of curves and the solid squares are morphisms of curves (i.e. fibre products). Then the family $\mathfrak{X} \rightarrow \mathcal{S}$ is smooth iff for all such A and I with a diagram above, we induce the dotted morphism

of families



Now suppose $\pi : \mathfrak{X} \rightarrow \mathcal{S}$ is étale over M . Then for all curves $p : C \rightarrow \operatorname{Spec} k$, $\operatorname{Isom}(\pi, p)$ must be étale over $\operatorname{Spec} k$. But this implies that $\operatorname{Isom}(\pi, p)$ is a finite set of reduced points. As each point in $\operatorname{Isom}(\pi, p)$ represents an isomorphism of C to $\pi^{-1}(s)$ (pulling back along $x \in \operatorname{Isom}(\pi, p)$ pulls back the isomorphism). Therefore

Proposition 2.19. *If $\pi : \mathfrak{X} \rightarrow \mathcal{S}$ is étale over M , then every curve C occurs a finite number of times in $\mathfrak{X}$. Furthermore, if C occurs in $\mathfrak{X}$ at all, it must have a finite automorphism group.*

Proposition 2.20. *A family of curves π is smooth over M and has property [Proposition 2.19](#) iff π is étale over M .*

Proof. Let $\pi' : \mathfrak{X}' \rightarrow \mathcal{S}'$ be another family of curves. By hypothesis, $\operatorname{Isom}(\pi, \pi') \rightarrow \mathcal{S}'$ is smooth. Then by [\[Stacks, Tag 02GU\]](#) and flatness of smooth maps, it suffices to show that the fibre over closed points has a finite number of closed points. But the set of closed points corresponds to how many times a fixed curve occurs in $\mathfrak{X}'$, so by hypothesis this is finite.

The converse was shown in the discussion above. $\square$

We naturally want to only consider families of curves étale over M in our topology, as our morphisms and coverings also have to be étale. Hence we define

Definition 2.21. A **modular family of curves** is a family of curves étale over M .

These families have nice properties:

Proposition 2.22. *Given two modular families of curves $\pi_i : \mathfrak{X}_i \rightarrow \mathcal{S}_i$ with a curve C occurring in π_i over the point s_i , the following holds:*

$$\hat{\mathfrak{o}}_{s_1} \cong \hat{\mathfrak{o}}_{s_2}.$$

Proof. Fix an isomorphism $\tau : \pi_1^{-1}(s_1) \rightarrow \pi_2^{-1}(s_2)$. Then τ determines a point $t \in \operatorname{Isom}(\pi_1, \pi_2)$ lying over s_1, s_2 . Since π_1, π_2 are moduli, $\operatorname{Isom}(\pi_1, \pi_2)$ is étale over both $\mathcal{S}_1, \mathcal{S}_2$. Thus we have isomorphisms

$$\hat{\mathfrak{o}}_{s_1} \cong \hat{\mathfrak{o}}_t \cong \hat{\mathfrak{o}}_{s_2}.$$

$\square$

Proposition 2.23. *Given a morphism $F : (\pi' : \mathfrak{Y} \rightarrow T) \rightarrow (\pi : \mathfrak{X} \rightarrow \mathcal{S})$ between two modular families of curves, F is étale for free (a priori a morphism of families has no extra properties).*

Proof. Since F is a morphism of families, $\mathfrak{Y}$ is isomorphic to $T \times_{\mathcal{S}} \mathfrak{X}$. Let the map on the bases be f . Therefore we have a map $\sigma : T \rightarrow \operatorname{Isom}(\pi', \pi)$ (note that this

is the base of the fibred product $\pi' \times_{\pi} \pi$). Thus we have the following diagram by the universal property of Isom .

$$\begin{array}{ccc} & \text{Isom}(\pi', \pi) & \\ \sigma \nearrow & & \searrow p_2 \\ T & \xrightarrow{f} & S \\ p_1 \nwarrow & & \nearrow \end{array}$$

By construction, this diagram commutes and $p_1\sigma = \text{id}_T$. Since p_1 is étale and the identity is étale, by properties of étale maps, σ is étale. Therefore f is too, completing the proof. $\square$

So our final definition of the moduli topology of families of elliptic curves is:

Definition 2.24. Let the topology $\mathfrak{M}$ be defined by

- (a) Open sets: families of elliptic curves and a final object M
- (b) Morphisms: morphisms of families of elliptic curves (étale by [Proposition 2.23](#)) and projections onto M
- (c) Covers: given $\{U_{\alpha} \xrightarrow{\pi_{\alpha}} U\}$
 - if $U \neq M$, it is a cover if $U = \cup \pi_{\alpha}(U_{\alpha})$
 - if $U = M$, it is a cover if every elliptic curve occurs in one of the families π_{α}

Remark 2.25. There is an alternative characterization of $\mathfrak{M}$ using the j -invariant and double covers, see [\[Mum65, p. 62\]](#) for the details.

There are a few standard sheaves associated to the topology $\mathfrak{M}$:

Definition 2.26. Let $\mathfrak{o}$ be the sheaf on $\mathfrak{M}$ sending $\pi : \mathfrak{X} \rightarrow \mathcal{S}$ to $\Gamma(\mathcal{S}, \mathfrak{o}_{\mathcal{S}})$. This can be easily verified to be a sheaf on $\mathfrak{M}$ without the final object. Then to define $\mathfrak{o}(M)$, we use the defining property of a sheaf: take a covering $\{\pi_{\alpha} : \mathfrak{X}_{\alpha} \rightarrow \mathcal{S}_{\alpha}\}$. Let $\pi_{\alpha, \beta} : \mathfrak{X}_{\alpha, \beta} \rightarrow \mathcal{S}_{\alpha, \beta}$ be the product of $\pi_{\alpha}, \pi_{\beta}$. Then let

$$\mathfrak{o}(M) := \ker\left(\prod \mathfrak{o}(\pi_{\alpha}) \rightarrow \prod \mathfrak{o}(\pi_{\alpha, \beta})\right)$$

where the map is the usual map (via restrictions to the image) $\Gamma(\mathcal{S}_{\alpha}, \pi_{\alpha}, \mathfrak{o}_{\mathcal{S}_{\alpha}, \pi_{\alpha}}) \rightarrow \prod \Gamma(\mathcal{S}_{\alpha, \beta}, \pi_{\alpha, \beta}, \mathfrak{o}_{\mathcal{S}_{\alpha, \beta}, \pi_{\alpha, \beta}})$.

Let $\mathfrak{o}^{\times}(\pi) = \text{units of } \mathfrak{o}(\pi)$.

Let $K(\pi)$ be the fraction field of $\mathfrak{o}(\pi)$.

Remark 2.27. Our approach is different to texts like [\[Hai14\]](#) (who does prove [Theorem 4.1](#)) in that we, following Mumford, construct the “universal family of elliptic curves” formally while Hain takes a more analytic approach through orbifolds. Despite these differences, underlying both texts are stacky ideas.

3. THE PICARD GROUP

Vector bundles are of great interest in math because they allow us to linearly approximate spaces, classify spaces, provide structure on spaces, and have relevance in ODEs. Vector bundles in general are quite tricky, so we can try classifying one dimensional ones. Vector bundles are analogous to sheaves, and one dimensional ones are analogous to invertible sheaves. By localizing at every point, we can even think of an invertible sheaf as a set of function from the space to a ring. Hence we define what an invertible sheaf on $\mathfrak{M}$:

Definition 3.1. An invertible sheaf on $\mathfrak{M}$ consists of the following:

- (a) for every $\pi : \mathfrak{X} \rightarrow \mathcal{S}$, an invertible sheaf $L(\pi)$ on $\mathcal{S}$
- (b) for all morphisms F between families:

$$\begin{array}{ccc} \mathfrak{X}_1 & \longrightarrow & \mathfrak{X}_2 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \mathcal{S}_1 & \xrightarrow{f} & \mathcal{S}_2 \end{array}$$

we have an isomorphism $L(F) : L(\pi_1) \cong f^*(L(\pi_2))$.

- (c) The isomorphisms have a compatibility condition: given two morphisms of families F, G ,

$$\begin{array}{ccccc} \mathfrak{X}_1 & \longrightarrow & \mathfrak{X}_2 & \longrightarrow & \mathfrak{X}_3 \\ \pi_1 \downarrow & & \pi_2 \downarrow & & \downarrow \pi_3 \\ \mathcal{S}_1 & \xrightarrow{f} & \mathcal{S}_2 & \xrightarrow{g} & \mathcal{S}_3 \end{array}$$

We have that the following diagram commutes:

$$\begin{array}{ccccc} & & f^*L(\pi_2) & & \\ & \nearrow \sim & \searrow \sim & & \\ L(\pi_1) & & L(F) & & f^*L(G) \\ & \searrow \sim & & & \nearrow \sim \\ & & (g \circ f)^*L(\pi_3) & & f^*(g^*L(\pi_3)) \end{array}$$

Remark 3.2. This information is sufficient to determine the value of the sheaf on the final object too, by letting $L(M)$ be the equalizer of

$$\prod_{U_\alpha \in \mathfrak{M}} L(U_\alpha) \rightrightarrows \prod_{U_\alpha, U_\beta \in \mathfrak{M}} L(U_\alpha \times U_\beta).$$

Example 3.3. We have an invertible sheaf Λ on $\mathfrak{M}$ defined on family $\pi : \mathfrak{X} \rightarrow \mathcal{S}$ by $\Lambda(\pi) := R^1\pi_*\mathcal{O}_{\mathfrak{X}}$ (where the R indicates the first right derived functor so that this is a sheaf on $\mathcal{S}$). We can then show that $\Lambda(\pi)$ is a locally free sheaf of rank 1 by verifying this on the stalks (Exercise 5.7 of Chapter II Hartshorne). On the stalks, $\Lambda(\pi)$ is the right derived functor of global sections of the structure sheaf of an elliptic curve $\pi^{-1}(s)$, which is free of rank 1. I.e., this is really just $H^1(C, \mathcal{O}_C)$. Furthermore, for all

$$\begin{array}{ccc} \mathfrak{X}_1 & \longrightarrow & \mathfrak{X}_2 \\ \downarrow & & \downarrow \\ \mathcal{S}_1 & \xrightarrow{f} & \mathcal{S}_2 \end{array}$$

we have a canonical isomorphism $\Lambda(\pi_1) \rightarrow f^*(\Lambda(\pi_2))$.

Definition 3.4. The **Picard group** $\text{Pic}(\mathfrak{M})$ is the group of isomorphism classes of invertible sheaves with the operation being via tensoring.

There is a conceptually cleaner way to define the Picard group using cohomology:

Theorem 3.5. $\text{Pic}(\mathfrak{M}) \cong H^1(\mathfrak{M}, \mathcal{O}^\times)$

But we will need the following theorem that allows us to think about invertible sheafs on a cover as a Zariski sheaf, thereby “descending” the topology to a topologically open set.

Theorem 3.6 (Hilbert-Grothendieck Theorem 90). *Let $\{U_\alpha \xrightarrow{q_\alpha} X\}$ be a flat covering of X . First suppose that for all α , we are given L_α , an invertible sheaf on U_α . Finally, suppose that for all α, β , we have an isomorphism $\phi_{\alpha, \beta}$ of $p_1^* L_\alpha$ and $p_2^* L_\beta$ on $U_\alpha \times_X U_\beta$ (with compatibility on triple products). Then we have an invertible sheaf L on X and isomorphisms $\psi_\alpha : L_\alpha \rightarrow q_\alpha^* L$ such that on $U_\alpha \times_X U_\beta$,*

$$\begin{array}{ccc} p_1^*(L_\alpha) & \xrightarrow[\phi_{\alpha, \beta}]{\sim} & p_2^*(L_\beta) \\ p_1^* \psi_\alpha \downarrow & & p_2^* \psi_\beta \downarrow \\ p_1^*(q_\alpha^*(L)) & \xlongequal{\quad} & p_2^*(q_\beta^*(L)) \end{array}$$

commutes (here the $\sim$ indicates an isomorphism).

Furthermore, L, ψ_α are uniquely determined up to canonical isomorphism.

Remark 3.7. This is related to the elementary statement of Hilbert Theorem 90 by being stronger than it. For a proof of this, see [Theorem B.1](#)

With this theorem, we will now prove the equivalence between the definitions. The following proof will be similar to the proof in the remark above, but more abstract.

Proof. First we construct a map $\text{Pic}(\mathfrak{M}) \rightarrow H^1(\mathfrak{M}, \mathfrak{o}^\times)$. Let L be an invertible sheaf on $\mathfrak{M}$. Then take an open cover $\{\pi_\alpha : \mathfrak{X}_\alpha \rightarrow \mathcal{S}_\alpha\}$ of M .

Now take some π_α . Because L is an invertible sheaf, we can replace the cover by one in which L is trivialized on the base (i.e. a particular Zariski open cover). Thus we can pick ϕ_α such that it is an isomorphism

$$L(\pi_\alpha) \xrightarrow[\phi_\alpha]{\sim} \mathfrak{o}_{\mathcal{S}_\alpha}.$$

Now let $\pi_{\alpha, \beta} : \mathfrak{X}_{\alpha, \beta} \rightarrow \mathcal{S}_{\alpha, \beta}$ and p_1, p_2 be the projections of $I_{\alpha, \beta} := \text{Isom}(\pi_\alpha, \pi_\beta)$ onto $\mathcal{S}_\alpha, \mathcal{S}_\beta$ respectively. Hence we have isomorphisms

$$p_1^* L(\pi_\alpha) \cong L(\pi_{\alpha, \beta}) \cong p_2^* L(\pi_\beta).$$

Thus we have the following:

$$\begin{aligned} \mathfrak{o}(I_{\alpha, \beta}) &= p_1^* \mathfrak{o}(\mathcal{S}_\alpha) \\ &\cong p_1^* L(\pi_\alpha) && \text{(via } \phi_\alpha) \\ &\cong L(\pi_{\alpha, \beta}) \\ &\cong p_2^* L(\pi_\beta) \\ &\cong p_2^* \mathfrak{o}(\mathcal{S}_\beta) && \text{(via } \phi_\beta^{-1}) \\ &= \mathfrak{o}(I_{\alpha, \beta}) \end{aligned}$$

(recall [Definition 2.26](#)) This isomorphism happens through multiplication by a unit $\sigma_{\alpha, \beta}$, hence we get an element of $\Gamma(I_{\alpha, \beta}, \mathfrak{o}_{I_{\alpha, \beta}}^\times) = \mathfrak{o}^\times(\pi_{\alpha, \beta})$. This then forms a 1-Cech cocycle in the sheaf $\mathfrak{o}^\times$ for this cover, call it λ_1 . We can verify that its derivative is zero by using the compatibility condition on morphisms in [Definition 3.1](#).

Next, we need to verify that this is well-defined, i.e. that $[\lambda_1] \in H^1(\mathfrak{M}, \mathfrak{o}^\times)$ is not dependent on the choice of $\{\phi_\alpha\}$. So suppose we have ϕ'_α , another choice of isomorphisms. Then $\phi'_\alpha = \sigma_\alpha \phi_\alpha$ for $\sigma_\alpha \in \Gamma(\mathcal{S}_\alpha, \mathfrak{o}_{\mathcal{S}_\alpha}^\times)$. Doing so gives us

$$\sigma'_{\alpha,\beta} = p_1^*(\sigma_\alpha) p_2^*(\sigma_\beta)^{-1} \sigma_{\alpha,\beta},$$

a homologous cocycle.

Thus λ_1 induces an element λ_2 of the Čech cohomology group. Now to show that λ_2 is independent of the cover we pick, given two covers, they are dominated by a cover finer than both. Thus we can assume the new cover is finer than $\{\pi_\alpha\}$. Hence the new λ_1 is a restriction of the old λ_1 , showing that λ_2 doesn't depend on the cover.

To see surjectivity, suppose we have an element $\lambda \in H^1(\mathfrak{M}, \mathfrak{o}^\times)$. This is induced by $\lambda_1 = \{\sigma_{\alpha,\beta}\}$ for some 1-Čech cocycle with cover of the final object $\{\pi_\alpha : \mathfrak{X}_\alpha \rightarrow \mathcal{S}_\alpha\}$. Now take $\pi : \mathfrak{X} \rightarrow \mathcal{S}$ and let $I_\alpha = \text{Isom}(\pi, \pi_\alpha)$. We want to use this cover and the data of λ_1 to construct an invertible sheaf on S .

First notice that $\{I_\alpha \rightarrow S\}$ is an étale cover of S . Then because we have projection maps $I_\alpha \times_S I_\beta \rightarrow I_\alpha \rightarrow S_\alpha$ and $I_\beta \rightarrow S_\beta$, we have a map

$$I_\alpha \times_S I_\beta \rightarrow \text{Isom}(\pi_\alpha, \pi_\beta).$$

Thus we induce a 1-Čech cocycle for the covering $\{I_\alpha \rightarrow S\}$, $\{\tau_{\alpha,\beta}\}$. We want to use this information to construct a sheaf on S . This is exactly what [Theorem 3.6](#) allows us to do.

Namely, we use the theorem with $U_\alpha = I_\alpha$, $X = S$, $L_\alpha = \mathfrak{o}_{I_\alpha}$, and $\phi_{\alpha,\beta} = \tau_{\alpha,\beta}$. (recall that ϕ needs to be merely an isomorphism of sheaves, not rings). So we get a sheaf L on S , which we define to be $L(\pi)$. To construct $L(F)$, just recall that morphisms of sheaves can be done locally and must satisfy exactly the compatibility condition. Finally, we must verify that L maps to λ_2 . This is obvious by construction of the $\phi_{\alpha,\beta}$ and picking $\{\pi_{S,\alpha} \times \pi_{S,\beta}\}$ to be the cover of M .

The last step necessary for this theorem is to prove injectivity: suppose $\lambda_2 = 0$. Then $\lambda_1 = 0$ for a cover $\{\pi_\alpha\}$. I.e., $\sigma_{\alpha,\beta} = 1$. We want to construct an isomorphism $\psi(\pi) : L(\pi) \cong \mathfrak{o}_S$ for every family $\pi : \mathfrak{X} \rightarrow \mathcal{S}$ that is compatible in the obvious ways. Like before, we consider the cover

$$\{I_\alpha := \text{Isom}(\pi, \pi_\alpha) \xrightarrow{q_\alpha} S\}.$$

Since L is an invertible sheaf, we have isomorphisms $L(\pi_\alpha) \rightarrow \mathfrak{o}_{S_\alpha}$. Thus pulling back L to I_α gives us an isomorphism $L(\pi \times \pi_\alpha) \cong \mathfrak{o}_{I_\alpha}$. By definition of an invertible sheaf, we then have an isomorphism

$$\psi_\alpha : q_\alpha^* L(\pi) \cong \mathfrak{o}_{I_\alpha}.$$

Since $\sigma_{\alpha,\beta} = 1$, the following diagram commutes:

$$\begin{array}{ccc} p_1^*(q_\alpha^* L(\pi)) & \xrightarrow{p_1^*(\psi_\alpha)} & p_1^*(\mathfrak{o}_{I_\alpha}) \\ \parallel & & \parallel \\ p_2^*(q_\beta^* L(\pi)) & \xrightarrow[p_2^*(\psi_\beta)]{\sim} & p_2^*(\mathfrak{o}_{I_\beta}) \end{array}$$

Thus we have descent data for $U_\alpha = I_\alpha$, $X = S$, $L_\alpha = \mathfrak{o}_{I_\alpha}$, and $\phi_{\alpha,\beta} = 1$. So by uniqueness, we have a canonical isomorphism of L and $\mathfrak{o}_S$. The verification of the

compatibility goes as follows: We need $\psi(\pi)$ to have the property that for every

$$\begin{array}{ccc} \mathfrak{X}_1 & \longrightarrow & \mathfrak{X}_2 \\ \downarrow & & \downarrow \\ \mathcal{S}_1 & \longrightarrow & \mathcal{S}_2 \end{array}$$

a morphism $F : \pi_1 \rightarrow \pi_2$, the following diagram commutes:

$$\begin{array}{ccc} L(\pi_1) & \xrightarrow{L(F)} & f^*L(\pi_2) \\ \downarrow \psi(\pi_1) & & \downarrow \psi(\pi_2) \\ \mathfrak{o}_{\mathcal{S}_1} & \xlongequal{\quad} & f^*\mathfrak{o}_{\mathcal{S}_2} \end{array}$$

This is true because we can use the canonicity of the isomorphism from descent data to reduce the commutativity to being on a cover. Then these isomorphisms commute on a cover by definition of a morphism of sheaves. $\square$

4. CALCULATION OF PICARD GROUP

Assume $\text{char } k \neq 2, 3$. In this section we will compute the Picard group of $\mathfrak{M}$ and see that it is:

Theorem 4.1. $\text{Pic}(\mathfrak{M}) \cong \mathbf{Z}/12\mathbf{Z}$

This is quite surprising, as 12 wouldn't be expected to appear in elliptic curves a priori. Furthermore, in the case of $k = \mathbf{C}$, this is basically saying that the first cohomology of the universal family of tori (genus 1 curves) has torsion! This contrasts with the moduli space of spheres at the origin being $\mathbf{R}^+$, which is contractible and thus having zero cohomology.

First we will want to construct a map $\text{Pic}(\mathfrak{M}) \rightarrow \mathbf{Z}/12\mathbf{Z}$. To attach an invariant to $L \in \text{Pic}(\mathfrak{M})$, we can try studying automorphisms of it. We know that every elliptic curve has an order two automorphism in the form of $i : (x, y) \mapsto (x, -y)$ when in Weierstrass form (i.e. $P \mapsto -P$). This automorphism then glues together to an automorphism of every family of elliptic curve:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{i} & \mathfrak{X} \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{S} & \xrightarrow{\text{id}} & \mathcal{S} \end{array}$$

This automorphism then induces an order two automorphism of $L(\pi)$. Any such automorphism has to be multiplication by ± 1 in $\mathfrak{o}_{\mathcal{S}}^\times$. This is a continuous function, so it defines a function to $\{\pm 1\}$ on the set of connected components of $\mathcal{S}$. Clearly we then have a homomorphism $\text{Pic}(\mathfrak{M}) \rightarrow \mathbf{Z}/2\mathbf{Z}$. Similarly, by using the existence of elliptic curves with automorphism groups of order 4 and 6 (corresponding to $j = 0$ and $j = 1728$ respectively) [Sil08, Theorem 10.1], we get maps

$$\text{Pic}(\mathfrak{M}) \rightarrow \mathbf{Z}/4\mathbf{Z} \quad \text{and} \quad \text{Pic}(\mathfrak{M}) \rightarrow \mathbf{Z}/6\mathbf{Z}.$$

Let C_A and C_B be curves with automorphism groups of order 4 and 6 respectively and let $\pi_A : C_A \rightarrow \text{Spec } k$ and $\pi_B : C_B \rightarrow \text{Spec } k$ be the trivial families. Furthermore, pick generators g, h of $\text{Aut}(C_A), \text{Aut}(C_B)$ respectively. By noting that $g^2 = h^3 = i$, we can combine them into a map

$$\beta : \text{Pic}(\mathfrak{M}) \rightarrow \mathbf{Z}/12\mathbf{Z}.$$

Before we get into the proof of injectivity and surjectivity, we need to build up a little more theory of elliptic curves.

Example 4.2. There is an example of a modular family of curves that we have seen, namely $\mathfrak{X}_0 \rightarrow S_0$ from [Example 2.3](#). To see this, we need to show [Proposition 2.18](#) and [Proposition 2.19](#). The latter is trivial to see. And the former can be shown with a rigorous analysis of infinitesimal deformations of an elliptic curve.

Observation 4.3. We can see that the j -line, i.e. $\mathbf{A}_j$ is not the base of a fine moduli space. By definition, a fine moduli space $\pi : \mathfrak{X} \rightarrow \mathcal{S}$ in $\mathfrak{M}$ will have every family mapping to it (since morphisms of families are pullback diagrams). Thus we have a map $\pi_0 \rightarrow \pi$. But by uniqueness of a map to a fine moduli space, $\pi_0 \rightarrow \pi$ must be the j -invariant map. Hence the question is whether we can extend $\mathfrak{X}_0$ to a family over $\mathbf{A}_j$.

So suppose we can extend it. Then by definition of π_M being modular, $\text{Isom}(\pi, \pi)$ must be étale over $\mathcal{S}$. By [Proposition 2.10](#), a closed point of $\text{Isom}(\pi, \pi)$ lying over a closed point $t \in \mathcal{S}$ is the data of

- (a) closed points t' such that there is an isomorphism $\pi^{-1}(t)$ and $\pi^{-1}(t')$
- (b) a choice of an isomorphism

But because we are in the j -line, there are no other closed points t' with an isomorphism to $\pi^{-1}(t)$. Thus the size of the fibres are the size of the isomorphism group. But elliptic curves with j -invariant 0 or 1728 don't have automorphism groups $\mathbf{Z}/2\mathbf{Z}$ like every other elliptic curve. Hence this map can't be étale, showing that an extension to a modular family doesn't exist.

Our second observation is that for any family of elliptic curves $\pi : \mathfrak{X} \rightarrow \mathcal{S}$, we have a morphism $j : \mathcal{S} \rightarrow \mathbf{A}_j$ by sending a point $s \in \mathcal{S}$ to $j(\pi^{-1}(s))$. This gives us a commutative diagram:

$$\begin{array}{ccc} \text{Isom}(\pi, \pi_0) & \xrightarrow{p_2} & S_0 \\ \downarrow p_1 & & \downarrow \\ \mathcal{S} & \xrightarrow{j} & \mathbf{A}_j \end{array}$$

The diagram commutes because closed points of $\text{Isom}(\pi, \pi_0)$ are isomorphisms of the elliptic curves lying over their images, hence they have the same j -invariant and thus lie over the same point in $\mathbf{A}_j$.

Observation 4.4. By [Observation 4.3](#), $S_0 \rightarrow \mathbf{A}_j$ is an étale double cover, hence $\text{Isom}(\pi, \pi_0)$ is an étale double cover of $j^{-1}(S_0) \subseteq \mathcal{S}$. This extends uniquely to a double cover I of all of $\mathcal{S}$ which is not necessarily étale!³ This double cover and map into $\mathbf{A}_j$ will allow us to formulate an equivalent definition of $\mathfrak{M}$. See [\[Mum65, p. 60\]](#) for the details.

Proof of Theorem 4.1. We have two parts: surjectivity and injectivity.

For surjectivity, I will show that $\beta(\Lambda)$ (where Λ is the invertible sheaf of [Example 3.3](#)) spans $\mathbf{Z}/12\mathbf{Z}$. To do so, we need to show that $\Lambda(\sigma)$ and $\Lambda(\tau)$ are the multiplication by a *primitive* root of unity maps. I.e., we need to show that

³a double cover here means a non-singular curve T and a finite flat surjective morphism $T \rightarrow S$ of degree two étale over an open dense subset of S . Either I is the disjoint union of two copies of S or $\text{Isom}(\pi, \pi_0)$ is the normalization of $j^{-1}(S_0)$ in a quadratic extension of its function field so that I is the normalization of S in this field.

$\text{Aut}(C_A), \text{Aut}(C_B)$ acts faithfully on $\Lambda(\pi_A), \Lambda(\pi_B)$ respectively. We do so concretely: by definition,

$$\Lambda(\pi_A) = H^1(C_A, \mathfrak{o}_{C_A}) \quad \text{and} \quad \Lambda(\pi_B) = H^1(C_B, \mathfrak{o}_{C_B}).$$

- (a) By Serre duality, $H^1(C, \mathfrak{o}_C)$ is isomorphic to the space of regular differentials on C
- (b) In Weierstrass form, the differential $\frac{dx}{y}$ is regular and forms a basis of the space of differentials on C
- (c) (i) $C_A : y^2 = x(x+1)(x-1)$ with g taking x to $-x$ and y to iy . Then $\frac{dx}{y} \mapsto \frac{-dx}{iy}$ so the action of $\text{Aut}(C_A)$ is faithful.
(ii) $C_B : y^2 = x^3 - 1$ with h taking x to ωx and y to $-y$. Then $\frac{dx}{y} \mapsto \frac{\omega dx}{-y}$ so the action of $\text{Aut}(C_B)$ is faithful.

Thus we are done with surjectivity.

To see injectivity, suppose $\beta(L) \equiv 0 \pmod{12}$. Let $\pi : \mathfrak{X} \rightarrow \mathcal{S}$ be a family of curves such that every curve occurs in it. Then $j : \mathcal{S} \rightarrow \mathbf{A}_j$ is a covering map. We shall descend our invertible sheaf on $\mathfrak{M}$ to an invertible sheaf L_0 on $\mathbf{A}_j$ via [Theorem 3.6](#). As there is a unique invertible sheaf on $\mathbf{A}_j$, we can then use this isomorphism to get an isomorphism $L \cong \mathfrak{o}$.

For this latter step, we can see this by taking an open cover on which L_0 is free U_i . By taking a distinguished affine subcover, we can suppose that $U_i = \text{Spec } k[j][u_i^{-1}]$ and $L_0|_{U_i} \cong \mathcal{O}_{k[j][u_i^{-1}]}$ (in a way compatible on intersections). So we have compatible isomorphisms on an open cover of $\mathbf{A}_j$. We can then glue them together to get an isomorphism $L_0 \cong \mathfrak{o}_{k[j]}$, showing uniqueness of invertible sheaf on $\mathbf{A}_j$.

Label maps as follows:

$$\begin{array}{c} \text{Isom}(\pi, \pi) \\ \downarrow f \\ \text{Normalization of } \mathcal{S} \times_{\mathbf{A}_j} \mathcal{S} \\ \downarrow g \\ \mathcal{S} \times_{\mathbf{A}_j} \mathcal{S} \\ \begin{array}{ccc} & \downarrow & \\ p_1 \swarrow & \mathcal{S} & \searrow p_2 \end{array} \end{array}$$

and let $q_i = p_i \circ g \circ f$ (which by observation are also the projection maps). In order to apply [Theorem 3.6](#) to conclude that L descends to $\mathbf{A}_j$, we need to show that the isomorphism $\psi : q_1^* L(\pi) \rightarrow q_2^* L(\pi)$ by virtue of L being an invertible sheaf is actually induced from an isomorphism $\psi_0 : p_1^* L(\pi) \rightarrow p_2^* L(\pi)$ via $f^* g^*$. This is because this will show that a sheaf on $\mathfrak{M}$ uniquely determines a sheaf on $\mathbf{A}_j$. Recall that by [Observation 4.4](#), $\text{Isom}(\pi, \pi) \rightarrow \text{Normalization of } \mathcal{S} \times_{\mathbf{A}_j} \mathcal{S}$ is a étale double cover.

Now take a closed point $\{\bar{t}\}$ of $\mathcal{S} \times_{\mathbf{A}_j} \mathcal{S}$ and let t_1, t_2 be two points of $\text{Isom}(\pi, \pi)$ lying over $\mathcal{S} \times_{\mathbf{A}_j} \mathcal{S}$. Let $\bar{L}_i$ be $p_i^* L$ restricted to $\bar{t}$ and $s_i = p_i(\bar{t})$. Then $\bar{L}_i \cong L(\pi^{-1}(s_i) \rightarrow s_i)$.

By [Proposition 2.10](#), t_1, t_2 represent isomorphisms τ, τ' of $\pi^{-1}(s_1)$ and $\pi^{-1}(s_2)$. By hypothesis, $\beta(L) \equiv 0$, so every automorphism of an elliptic curve induces a

trivial automorphism of L . Thus $L(\tau' \circ \tau^{-1}) = \text{id}$. Hence $L(\tau') = L(\tau)$. Hence ψ induces a unique isomorphism ψ_0 of $p_1^*L(\pi)$ and $p_2^*L(\pi)$ at $\bar{t}$. Showing that this topological map is given by functions in the local rings of $\mathcal{S} \times_{\mathbf{A}_j} \mathcal{S}$ is pretty technical and will be skipped.

Then the ψ_0 forms descent data for $\{\mathcal{S} \rightarrow \mathbf{A}_j\}$ with the compatibility condition following from the compatibility conditions on being an invertible sheaf. Thus we have unique descent data, thereby showing that L determines and is determined by an invertible sheaf on $\mathbf{A}_j$. As there is only one invertible sheaf on $\mathbf{A}_j$, the structure sheaf, we are done. $\square$

One way to think about this result is that $\text{Pic}(\mathfrak{M})$ consists of global line bundles on all elliptic curves. Then these line bundles are heavily controlled by the line bundles point wise on $\mathfrak{M}$ (i.e. line bundles on curves) as they must factor in the automorphisms of the curves.

APPENDIX A. TENSORS

Proposition A.1. *Let L/K be a finite Galois extension of fields with Galois group G . Then*

$$L \otimes_K L \cong \prod_{\sigma \in G} L.$$

Proof. By the primitive element theorem, write $L \cong K(\theta)$ with the minimal polynomial of θ being $P \in K[X]$. Then

$$L \otimes_K L \cong L \otimes_K (K[X]/P) \cong (L \otimes_K K[X])/P \cong L[X]/P.$$

Since L is Galois over K , P factors into $\prod_{\sigma \in G} (X - \sigma\theta)$. Thus by the Chinese remainder theorem,

$$L[X]/P \cong \prod L[X]/(X - \sigma\theta).$$

$\square$

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Theorem B.1. *Theorem 3.6 implies that for a Galois extension L/K with cyclic Galois group G , any element a with relative norm 1 is equal to*

$$\frac{b}{\sigma b}$$

for some $b \in L$.

Proof. First note that the statement about norm 1 elements implies that for all finite Galois extensions L/K ,

$$H^1(\text{Gal}(L/K), L^\times) = \{1\}.$$

Now fix L/K a finite Galois extension and consider the cover $\{\text{Spec } L \rightarrow \text{Spec } K\}$. Take an L -vector space V on $\text{Spec } L$ and let $\mathcal{V}$ be the sheaf on $\text{Spec } L$ associated to V . Fix e a basis vector of V . Then to define $\phi_{\alpha, \beta}$, it suffices to define the map on global sections (i.e. $(L \otimes_K L) \otimes V$) and to define it on $1 \otimes_K 1 \otimes e$ since we can linearly extend (note that here, when we linearly scale e by $\ell \in L$, we have to scale the image by $L \otimes 1$ by $L \otimes_K L$ linearity). Then given $[\lambda] \in H^1(\text{Gal}(L/K), L^\times)$, we can construct descent data as follows:

Lemma B.2. *Our first claim will be that descent data bijects to 1-cocycles.*

Proof. Observe that $L \otimes_K L \cong \prod_{\sigma \in G} L$ as rings (Proposition A.1). Thus an isomorphism $p_1^*V \rightarrow p_2^*V$ is the same as giving a unit $\ell \in (\prod_{\sigma \in G} L)^\times$.

Now suppose we have descent data. Then we have a unit $\ell \in (\prod_{\sigma \in G} L)^\times$. By the identification above, we then have a function $f : G \rightarrow L^\times$ by looking at each coordinate. It is a 1-cocycle because the cocycle condition for descent data implies that

$$\begin{array}{ccc} p_1^*V & \xrightarrow{p_{13}^*\phi_{\alpha,\beta}} & p_3^*V \\ & \searrow p_{12}^*\phi_{\alpha,\beta} & \nearrow p_{23}^*\phi_{\alpha,\beta} \\ & p_2^*V & \end{array}.$$

Note that here, the single digit projection p_i denotes projection from $L_1 \otimes_K L_2 \otimes_K L_3$ (labelled for convenience) to L_i , while p_{ij} is projection from $L_2 \otimes_K L_2 \otimes_K L_3$ to $L_i \otimes_K L_j$. By Proposition A.1, scaling via $p_1 : L \rightarrow L \otimes_K L$ ($\ell \otimes 1$) multiplies the image in $\prod_{\sigma} L$ by ℓ in each coordinate, while scaling via p_2 ($1 \otimes \ell$) multiplies the σ -th coordinate by $\sigma\ell$.

Now applying Proposition A.1 again, we have that $L \otimes_K L \otimes_K L \cong \prod_{\sigma} \prod_{\sigma\tau} L$ (the strange indexing inside is because we factor Q , the minimal polynomial of the primitive element of the second L into terms of X). Thus computing the compositum path by converting everything via Proposition A.1, we first multiply by $((\lambda(\sigma))_{\sigma\tau})_{\sigma}$ and then multiply by $((\sigma\lambda(\tau))_{\sigma\tau})_{\sigma}$. Computing the map $p_1^*V \rightarrow p_3^*V$, we get multiplication by $((\lambda(\sigma\tau))_{\sigma\tau})_{\sigma}$, exactly as desired.

Reversing the above proof gives the opposite direction. $\square$

Lemma B.3. *Our next claim is that isomorphisms of descent data correspond exactly to cohomologous 1-cocycles.*

Proof. An isomorphism of descent data is an isomorphism $f : V \rightarrow V$ such that the following commutes:

$$\begin{array}{ccc} p_1^*V & \xrightarrow{\phi_{\alpha,\beta}} & p_2^*V \\ \downarrow p_1^*f & & \downarrow p_2^*f \\ p_1^*V & \xrightarrow{\phi'_{\alpha,\beta}} & p_2^*V \end{array}.$$

We know that $\phi_{\alpha,\beta}$ is really multiplication by some unit u in $\prod_{\sigma} L$. An isomorphism f is just multiplication by some $\ell \in L^\times$, so we have the following diagram:

$$\begin{array}{ccc} p_1^*V & \xrightarrow{\phi_{\alpha,\beta}} & p_2^*V \\ \downarrow p_1^*(\cdot\ell) & & \downarrow p_2^*(\cdot\ell) \\ p_1^*V & \xrightarrow{\phi'_{\alpha,\beta}} & p_2^*V \end{array}.$$

In the above lemma we described what these two vertical maps were. Thus the commutativity of this diagram is saying that in $\prod_{\sigma} L$ (and denoting the σ -th coordinate of u by u_{σ})

$$(\ell u'_{\sigma})_{\sigma} = (\sigma \ell u_{\sigma})_{\sigma}.$$

This is saying that $u'(\sigma) = u_{\sigma}$ and $u(\sigma) = u(\sigma)$ are cohomologous.

To complete the proof, just reverse the direction. $\square$

Finally, [Theorem 3.6](#) implies that up to isomorphism, there is only one descent data, coming from the K vector space K . Thus there is only one cohomology class, implying Hilbert Theorem 90. $\square$

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